# Lecture 3 The axioms of set theory

The alphabet we shall use to express Set Theory is described by the following table of symbols.

| | The alphabet of Set Theory |
|---|---|
| parentheses | ( , ) |
| connectives & quantifiers | $\Leftrightarrow$ , $\Rightarrow$ , $\vee$ , $\wedge$ , $\neg$ , $\forall$ , $\exists$ |
| variables | $x, y, z; x_1, y_1, z_1; x_2, y_2, z_2; \dots$ |
| constants | $\emptyset$ |
| atomic predicates | $\in$ , $=$ |

The axioms of Set Theory are the Zermelo - Fraenkel axioms together with the Axiom of Choice. We shall describe them as follows.

1. *Extensionality*

$$\forall x \left( \forall y \left( (x = y) \iff \forall z \left( z \in x \iff z \in y \right) \right) \right)$$

In other words, the sets are determined by its elements. We shall use this axiom a lot in our proofs, usually splitting the proof of an equality of sets into the "$\Longrightarrow$" case and the "$\Longleftarrow$" case separately.

## 2 Foundation

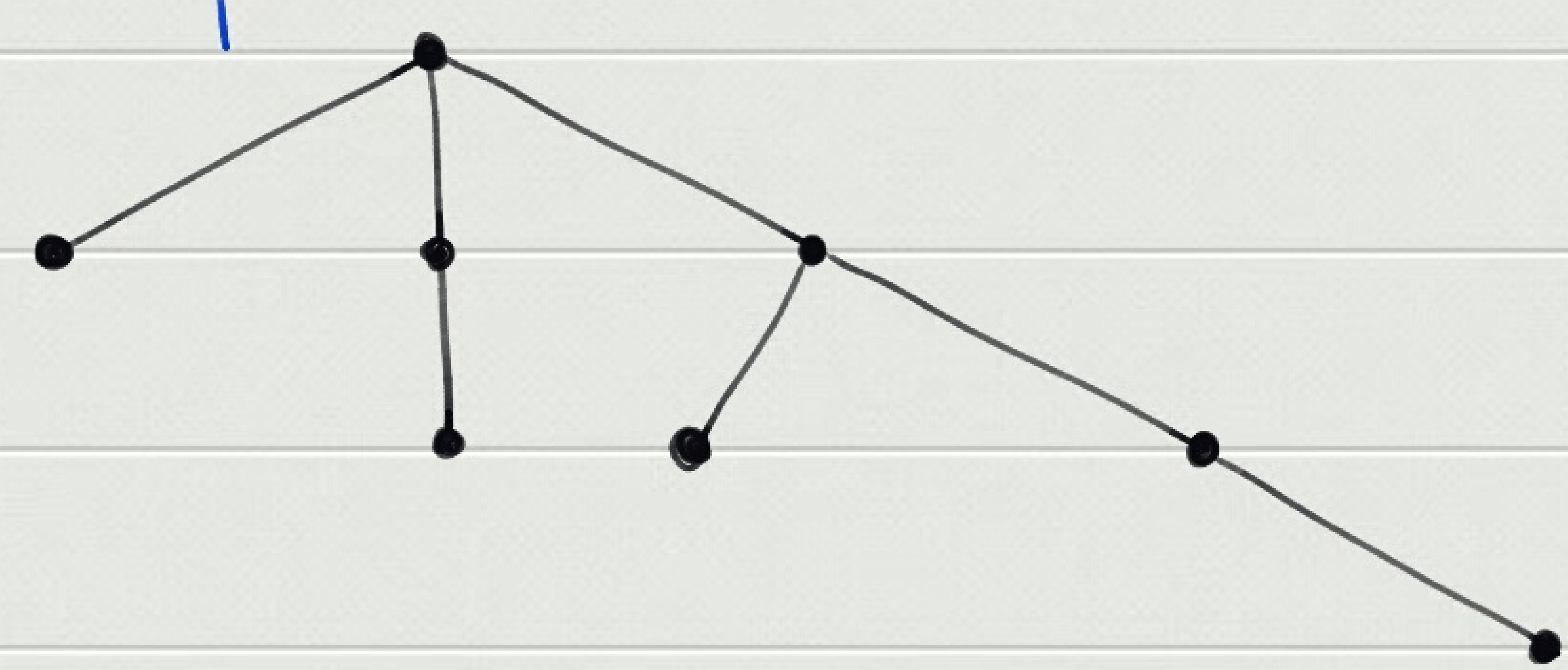
$$\forall x \, ( x \neq \phi \Rightarrow \exists y \, ( y \in x \text{ minimal} ) )$$

Here we say that an element $y \in x$ is *minimal* if for all $z \in x$ we have $z \notin y$.

So this axiom says all descending chains

$$\ldots \in x_3 \in x_2 \in x_1 \in x_0 \qquad \text{☆}$$

must terminate, i.e. given a set $x$, the branches of the tree determined by $\_ \in \_$ all have finite length. For example, the tree of the set $\{ \phi, \{\phi\}, \{\phi, \{\phi\}\} \}$ is



## 3. Paring

$$\forall x\,(\forall y\,(\exists z\,(z = \{x, y\})))$$

In particular, this means that given sets $x, y$,

$$(x, y) := \{\{x\}, \{x, y\}\}$$

is a set. Following Kuratowski, it defines *ordered pair* as the Extensionality Axiom implies that for all sets $x, y, x_1, y_1$, we have

$$(x, y) = (x_1, y_1)$$

if and only if $x = x_1$ and $y = y_1$.

## 4. Union

$$\forall x \left( \exists y \left( y = \bigcup_{z \in x} z \right) \right),$$

where $w \in \bigcup_{z \in x} z$ if and only if there is $z \in x$ s.t. $w \in z$.

For example, if $c := \{ \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\} \}$, then

$$\bigcup_{z \in c} z = \{ \emptyset, \{\emptyset\} \}$$

<u>Remark</u> Later we'll introduce the notion of an indexed family $\{X_i\}_{i \in I}$ of sets and define their union $\bigcup_{i \in I} X_i$.

5. *Infinity*

$$\exists x \left( \underbrace{\exists e (\forall y (\neg (y \in e)))}_{\text{existence of an empty set}} \wedge (e \in x) \wedge (\underbrace{\forall y (y \in x \Rightarrow S(y) \in x)}_{x \text{ is closed under } y \mapsto S(y)}) \right)$$

where $S(y) := y \cup \{y\}$ is the succesor of $y$. The Fundation Axiom implies that

$$y,\ S(y),\ (S \circ S)(y),\ \ldots$$

consists of distinct sets. Indeed, $y = S(y)$ means $y = y \cup \{y\}$, so by the Extensionality Axiom $y \in y$, which gives an infinite descending chain

$$\ldots \in x \in x \in x \in x$$

thus contradicting $(\star)$. Similarily, each of the equations

$$y = (S \circ S)(y), \ \ldots$$

gives again $y \in y$, which contradicts $(\star)$. In particular, the Infinity Axiom implies that we have the ascending chain of sets

$$0 \in 1 \in 2 \in \ldots$$

defined recursively by letting

$$0 := \emptyset \quad \text{and} \quad i+1 := S(i), \quad \text{if } i \neq 0.$$

Thus we get the infinite set

$$\omega := \{0, 1, 2, \dots\}$$

and a further ascending chain

$$\omega \in \omega+1 \in \omega+2 \in \cdots$$

by letting $\omega + 1 := S(\omega), \dots$ The *von Neumann ordinals*[1] are

$$0, 1, 2, \dots \quad \omega+1, \omega+2, \dots$$

---

1. von Neumann, J., *Zur Einführung der transfiniten Zahlen*, Acta Litterarum ac scientiarum Ragiae Universitatis Hungaricae Francisco-Josephinae, Sectio scientiarum matematicarum, 1, pp 199–208.

6. *Power set*

$$\forall x \left( \exists y \left( y = P(x) \right) \right)$$

Here $P(x) := \{ z \in y \mid z \subseteq x \}$ is the power set, where we define

$$(z \subseteq x) \Leftrightarrow \left( \forall w \left( w \in z \Rightarrow w \in x \right) \right),$$

known as the *subset relation*.

7. *Separation*

$$\forall x \left( \exists y \left( y = \{ z \in x \mid P(z) \} \right) \right)$$

where $P(z)$ is a predicate of the language of Set Theory.

8. Replacement

$$\forall x \left( \forall y \left( \text{the image of any function } x \xrightarrow{f} y \text{ is a set} \right) \right)$$

<u>Remark</u> This axiom is useful in the proof of existence of limit ordinals.

9. Choice axiom

$$\forall x_i \cdots \exists y \left( y = \prod_i x_i \right)$$