Lecture 4 Further remarks on Set Theory

Recall that from the Fundation axiom we get

$$\forall x \left(\neg \left(x \in x \right) \right)$$

Indeed, if

$$\exists x \left(x \in x \right)$$

then we get an infinite chain

$$\dots \in x \in x \in x \in x$$

Thus contradicting ($\star$). In Cantor's Set Theory, given any predicate P of the language of Set Theory, one is alowed to form the set $\{ x \mid P(x) \}$, so for Cantor

$$R := \{x \mid \neg(x \in x)\}$$

is a set. But we have

$$x \in R \iff \neg(x \in R),$$

a contradiction.

This argument is known as *Russell's Paradox*. The Zermelo-Fraenkel Set Theory addresses Russell's Paradox. Indeed, just combine the Separation Axiom and the Fundation Axiom; the set of all sets is not a set.

The universe of discourse of Set Theory is the von Neumann universe V, which is constructed inductively, starting from ϕ.

$$
\begin{array}{l}
V_0 := \phi \\
\cap\! \mid \\
V_1 := P(V_0) = \{\phi\} \\
\cap\! \mid \\
V_2 := P(V_1) = \{\phi, \{\phi\}\} \\
\cap\! \mid \\
\vdots
\end{array}
$$

Then we define

$$V_\omega := \bigcup_{i \in \omega} V_i$$

where $\omega = \{0, 1, 2, \dots\}$. Moreover, we let

$$\begin{array}{l} V_{\omega+1} := P(V_\omega) \\ \cap\!\! | \\ V_{\omega+2} := P(V_{\omega+1}) \\ \cap\!\! | \\ \vdots \end{array}$$

Recall that ω exists because of the Infinity Axiom. It is the first limiting ordinal.