Lecture 1 Bernoulli numbers and the zeta function

From Jakob Bernoulli, *Ars Conjectandi* (1713) we introduce the sequence $B_0, B_1, B_2, \ldots$ defined by the Taylor expansion

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$

We may compute the B_k's recursively, as follows.

Indeed, if we have power series

$$f = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$g = b_0 + b_1 x + b_2 x^2 + \cdots$$

then $fg = c_0 + c_1 x + c_2 x^2 + \cdots$, where

$$\left.\begin{aligned}
c_0 &= a_0 b_0 \\
c_1 &= a_0 b_1 + a_1 b_0 \\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\
c_3 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \ddots
\end{aligned}\right\} \bigstar$$

In particular, if $a_0 \neq 0$ then we may compute the power series of $g = \frac{1}{f}$ from $(\star)$ since

$$
\begin{aligned}
1 &= a_0 b_0 \\
0 &= a_0 b_1 + a_1 b_0 \\
0 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\
0 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \\
&\vdots \qquad \vdots \qquad \ddots
\end{aligned}
$$

implies that $b_0 = a_0^{-1}$ and recursively

$$
\begin{aligned}
b_1 &= -a_0^{-1}(a_1 b_0) \\
b_2 &= -a_0^{-1}(a_1 b_1 + a_2 b_0) \\
b_3 &= -a_0^{-1}(a_1 b_2 + a_2 b_1 + a_3 b_0) \quad \ddots \\
&\vdots \qquad \vdots
\end{aligned}
$$

Therefore, if

$$f = \frac{e^x - 1}{x} = \frac{1}{x}\left(-1 + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = 1 + \frac{1}{2}x + \frac{1}{6}x^2 + \cdots$$

then

$$\frac{1}{f} = \frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \ldots$$

So

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0,$$

$$B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{691}{2730}, \quad \ldots$$

<u>Lemma</u> For each $k \in \{3, 5, 7, \ldots\}$ we have

$$B_k = 0.$$

Proof

[Ex.] Hint: Consider $h = \dfrac{x}{e^x - 1} - 1 + \dfrac{1}{2}x$ show its even.

Remark Later we shall discuss Ramanujan's famous congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691},$$

where

$$\Delta := q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

and

$$\sigma_{11}(n) := \sum_{0<d \mid n} d^{11}.$$

Clearly the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely and uniformly on all compact subsets of $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 \}$. It defines the Riemann zeta function.

<u>Thm</u> (Euler) For each $k \in \mathbb{Z}_{\geq 1}$ we have

$$\zeta(2k) = (-1)^{k-1} \frac{B_{2k}}{2\,(2k)!} (2\pi)^{2k}.$$

Proof

Note that

$$\pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z} = \pi i \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = \pi i \left(1 + \frac{2}{e^{2\pi i z} - 1} \right)$$

$$= \pi i + \frac{2\pi i}{e^{2\pi i z} - 1} = \pi i + \frac{1}{z} \cdot \frac{2\pi i z}{e^{2\pi i z} - 1}$$

$$= \pi i + \frac{1}{z} \sum_{k=0}^{\infty} B_k \frac{(2\pi i z)^k}{k!} = \frac{1}{z} + \frac{1}{z} \sum_{k=2}^{\infty} B_k \frac{(2\pi i z)^k}{k!}$$

$$= \frac{1}{z} + \frac{1}{z} \sum_{k=1}^{\infty} B_{2k} \frac{(2\pi i z)^{2k}}{(2k)!},$$

Hence we have

$$\pi \cot \pi z = \frac{1}{z} \sum_{k=0}^{\infty} B_{2k} \frac{(2\pi i z)^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k} (2\pi)^{2k}}{(2k)!} z^{2k-1} . \qquad (1)$$

But we have the Laurent expansion

$$\pi \cot \pi z = \frac{1}{z} - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k-1} . \qquad (2)$$

Indeed, recall that for each open $\mathcal{U} \subseteq \mathbb{C}$ we have a group homomorphism

$$\begin{aligned} \mathcal{M}(\mathcal{U})^{\times} &\longrightarrow \mathcal{M}(\mathcal{U}) \\ f &\longmapsto \mathcal{D}(f) := \frac{f'}{f} \end{aligned}$$

from the multiplicative group of non-zero meromorphic functions $\mathcal{M}(\mathcal{U})^{\times}$ to the additive group $\mathcal{M}(\mathcal{U})$, i.e

$$\mathcal{D}(fg) = \mathcal{D}(f) + \mathcal{D}(g)$$

$\forall\, f, g \in \mathcal{M}(\mathcal{U})^{\times}$. Therefore the well-known infinite product

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right)$$

yields the series

$$\pi \cot \pi z = \frac{1}{z} + 2 \sum_{k=1}^{\infty} \frac{z}{z^2 - k^2} = \frac{1}{z} - 2 \sum_{k=1}^{\infty} \frac{z}{k^2} \cdot \frac{1}{1 - \frac{z^2}{k^2}}$$

$$= \frac{1}{z} - 2 \sum_{k=1}^{\infty} \frac{z}{k^2} \cdot \sum_{n=1}^{\infty} \left(\frac{z^2}{k^2} \right)^{n-1} = \frac{1}{z} - 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{z^{2n-2}\, z}{k^{2n-2}\, k^2}$$

$$= \frac{1}{z} - 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^{2n}} \cdot z^{2n-1} = \frac{1}{z} - 2 \sum_{n=1}^{\infty} \zeta(2n) z^{2n-1}.$$

By comparing (1) and (2) the theorem follows $\square$

<u>Remark</u> From the above we also get the series

$$f := \pi \frac{\cos \pi z}{\sin \pi z} = \pi \cot \pi z = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\frac{1}{z+n} - \frac{1}{n} \right).$$

Differentiating repeatedly f yields

$$\left.\begin{aligned} f^{(1)} &= - \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} \\ f^{(2)} &= 2 \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^3} \\ f^{(3)} &= -2 \cdot 3 \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^4} \\ &\vdots \end{aligned}\right\} \quad A$$

The formulae shall be applied to obtain the Fourier expansions of the Eisenstein series, such as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n,$$

$$q = e^{2\pi i \tau}, \qquad \tau \in \mathcal{H} := \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}.$$

See Marina Viazovska, *The sphere packing problem in dimension 8*. The above Eisenstein series is in fact the theta function attached to the lattice Γ_8.