Lecture 2 Another result from Ars Conjectandi

Define $\forall\, k, m \in \mathbb{N}$ the sum

$$S_k(m) = \sum_{n=1}^{m} n^k$$

Thm (Jakob Bernoulli) Notation as above,

$$S_k(m) = \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_i\, m^{k-i+1}$$

Proof

Consider the k-th Bernoulli polynomial,

$$B_k(X) = \sum_{i=0}^{k} \binom{k}{i} B_i X^{k-i} \in \mathbb{Q}[X]$$

and let

$$\mathbb{C} \times \mathbb{C} \xrightarrow{F} \mathbb{C}$$

$$(t, x) \mapsto t \frac{e^{t(1+x)}}{e^t - 1}$$

Note that

$$F(t,x) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \sum_{k=0}^{\infty} x^k \frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} B_i \, x^{k-i} \frac{t^k}{k!} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

But $F(t,x) - F(t, x-1) = t e^{tx}$, thus

$$B_{k+1}(x) - B_{k+1}(x-1) = (k+1)x^k$$

Hence

$$S_k(m) = \frac{1}{k+1} \left(B_{k+1}(m) - B_{k+1}(0) \right)$$

Euler used this theorem and his formula to attempt to make sense of the expression

$$1 - 2^r + 3^r - 4^r + \dots = \pm 2 \frac{r!}{\pi^{r+1}} \left(1 + \frac{1}{3^{r+1}} + \frac{1}{5^{r+1}} + \dots \right),$$

for $r = 1, 3, 5, 7, \dots$ (See p 274 of Weil's *Number Theory, An approach through history, From Hammurabi to Legendre*.) He anticipated the functional equation

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) := \frac{1}{2} \pi^{-\frac{s}{2}} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

We shall see that this makes possible to analytically extend the zeta function to $\mathbb{C} \setminus \{1\}$. Riemann conjectured that the non trivial zeroes of $\zeta(s)$ (i.e. apart from $s = -2, -4, -6, \dots$ explained via the theory of the Γ function) all lie on the line

$$\{ s \in \mathbb{C} \mid \operatorname{Re}(s) = \tfrac{1}{2} \}$$

This is known as the *Riemann Hypothesis*.

It turns out to be equivalent to the statement

For every $\varepsilon > 0$ there is b s.t. for all m :

$$b < m \implies \left| \sum_{n=1}^{m} \mu(n) \right| < m^{\frac{1}{2}+\varepsilon}$$

See p. 14 of Manin, A Course in Mathematical Logic for Mathematicians, Second Edition, With Collaboration by Boris Zilber, GTM 53, Springer Verlag.

Here $\mu(n)$ is the Möbius function,

$$\mu(n) := \begin{cases} (-1)^r, & \text{if } n = p_1 \cdot \ldots \cdot p_r, \text{ with primes} \\ & p_1 < p_2 < \cdots < p_r \\ 0, & \text{otherwise.} \end{cases}$$

As we shall see, Riemann's starting point is the following.

<u>Thm</u> (Euler) For $s \in \mathbb{C}$ s.t. $\mathrm{Re}(s) > 1$

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$