Lecture 3 Functional equation of the Riemann ζ-function

Defn For each $s \in \mathbb{C}$ s.t. $\mathrm{Re}(s) > 0$ let

$$\Gamma(s) := \int_0^\infty e^{-t}\, t^s \,\frac{dt}{t}$$

For cognoscenti

We may have $d\mu(t) = \frac{dt}{t}$, where μ is the unique (up to a positive multiplicative constant) translation-invariant Lebesgue measure on the locally compact topological group $G = \mathbb{R}^\times_{>0}$, known as the *Haar measure* of G.

Thm For all $s \in \mathbb{C}$ s.t. $\mathrm{Re}(s) > 0$

$$\Gamma(s+1) = s\,\Gamma(s).$$

Proof

[Ex. — just integrate by parts]

Thm The function $\Gamma(s)$ extends to a meromorphic function that is holomorphic on $\mathbb{C}$ except for $s = 0, -1, -2, -3, \ldots$

Proof

By recursively setting $\Gamma(s) := \dfrac{\Gamma(s+1)}{s}$ we arrive at the claimed meromorphic function $\square$

<u>Remark</u> From the functional eq^n for $\Gamma(s)$ and the fact that $\Gamma(1) = 1$, we may see that for all $n \in \mathbb{Z}_{>0}$

$$\Gamma(n) = (n-1)!$$

<u>Thm</u> (Riemann) Let $\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$. Then

$$\Lambda(s) = \Lambda(1-s). \qquad \bigstar$$

<u>Remark</u> The functional eq^n ($\bigstar$) gives us an extension of $\zeta(s)$ to all $z \in \mathbb{C}$ s.t. $\mathrm{Re}(s) < 0$. For the critical strip $0 \leqslant \mathrm{Re}(s) \leqslant 1$ the extension will be addressed separately.

We'll prove the above thm following the paper

Riemann, B., Ueber die Anzahl der Primzahlen unter einer gegeben Grösse, Monatsberichte der Berliner Akademie, 1859.

We shall need the following

<u>Defn</u> Given a function $f : \mathbb{R}_{>0} \longrightarrow \mathbb{C}$ of rapid decay, its *Mellin transform* Mf is

$$(Mf)(s) := \int_0^\infty f(t)\, t^s \, \frac{dt}{t}$$

Example If $f : \mathbb{R}_{>0} \longrightarrow \mathbb{C}$ is just $f(t) = e^{-t}$, then we have $\Gamma(s) = (Mf)(s)$.

Consider the theta function

$$\begin{aligned} \mathbb{R}_{>0} &\xrightarrow{\vartheta} \mathbb{C} \\ t &\longmapsto \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} \end{aligned}$$

This is not a function of rapid decay, but $\omega(t) := \dfrac{\vartheta(t) - 1}{2}$ has this nice property. Moreover, we have the following.

Thm The Mellin transform $M\omega$ of ω is

$$(M\omega)(s) = \pi^{-s}\Gamma(s)\zeta(2s) = \Lambda(2s)$$

Proof

$$(M\omega)(s) = \int_0^\infty \omega(t)\, t^s \frac{dt}{t} = \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t}\right) t^s \frac{dt}{t}$$

$$= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} \cdot t^s \frac{dt}{t}$$

$$= \sum_{n=1}^\infty \int_0^\infty e^{-u} \pi^{-s} n^{-2s} u^s \frac{du}{u} \qquad (u := \pi n^2 t)$$

$$= \pi^{-s} \left(\int_0^\infty e^{-u} \frac{du}{u}\right)\left(\sum_{n=1}^\infty n^{-2s}\right) = \pi^{-s}\Gamma(s)\zeta(2s)$$

$\square$

Thm For all $t \in \mathbb{R}_{>0}$

$$\vartheta\left(\frac{1}{t}\right) = \sqrt{t}\ \vartheta(t)$$

Proof

[We'll prove this thm next lecture — using Poisson sumation.]

Corollary For all $t \in \mathbb{R}_{>0}$: $\omega\left(\frac{1}{t}\right) = \frac{1}{2}\left(\sqrt{t} - 1\right) + \sqrt{t} \cdot \omega(t)$.

Proof

$$\omega\left(\frac{1}{t}\right) = \frac{\vartheta\left(\frac{1}{t}\right) - 1}{2} = \frac{\sqrt{t}\ \vartheta(t) - 1}{2} = \frac{\sqrt{t}\left(1 + 2\omega(t)\right) - 1}{2}$$

$$= \frac{1}{2}\left(\sqrt{t} - 1\right) + \sqrt{t} \cdot \omega(t) \qquad \square$$

Proof of Riemann's theorem

We have

$$\Lambda(s) = (M\omega)\left(\tfrac{s}{2}\right) = \int_0^\infty \omega(t)\, t^{\frac{s}{2}} \frac{dt}{t} =: I$$

$\forall$ $s \in \mathbb{C}$ s.t $\mathrm{Re}(s) > 1$ since

$$\omega(t) \approx C \cdot \frac{1}{\sqrt{t}}, \text{ as } t \to 0. \qquad (*)$$

[Prove (*) as an exercise.]

By braking down the above integral into two pieces we get

$$I = \underbrace{\int_0^1 \omega(t)\, t^{\frac{s}{2}} \frac{dt}{t}}_{\overset{t \mapsto \frac{1}{t}}{=} \; -\int_\infty^1 \omega\left(\frac{1}{t}\right) t^{-\frac{s}{2}} \frac{dt}{t}} + \int_1^\infty \omega(t)\, t^{\frac{s}{2}} \frac{dt}{t}$$

$$-\int_\infty^1 \omega\left(\frac{1}{t}\right) t^{-\frac{s}{2}} \frac{dt}{t} = \int_1^\infty \left(\sqrt{t}\,\omega(t) + \frac{\sqrt{t}}{2} - \frac{1}{2}\right) t^{-\frac{s}{2}} \frac{dt}{t}$$

Hence

$$\Lambda(s) = \int_1^\infty \omega(t)\, t^{\frac{1-s}{2}} \frac{dt}{t} + \int_1^\infty \omega(t)\, t^{\frac{s}{2}} \frac{dt}{t} + \underbrace{\frac{1}{2}\int_1^\infty t^{-\frac{1+s}{2}}\, dt - \frac{1}{2}\int_1^\infty t^{-1-\frac{s}{2}}\, dt}_{= -\frac{1}{1-s} - \frac{1}{s}}$$

Thus $\Lambda(1-s) = \Lambda(s)$ $\square$