Lecture 4 Poisson summation

Recall that, following Riemann, we introduced the function $\vartheta : \mathbb{R}_{>0} \longrightarrow \mathbb{C}$ defined by

$$\vartheta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$$

and used the identity

$$\vartheta\left(\frac{1}{t}\right) = \sqrt{t}\, \vartheta(t) \qquad \text{A}$$

to obtain the functional eq^{n} of the ζ-function. We shall prove (A) using tools from Fourier analysis.

<u>Defn</u> Given any $f \in L^2(\mathbb{R})$, its *Fourier transform* is

$$\hat{f} : \mathbb{R} \longrightarrow \mathbb{C}$$
$$s \longmapsto \int_{\mathbb{R}} e^{-2\pi i s t} f(t)\, dt$$

Note for the cognoscenti

For each G locally compact abelian group its *Pontryagin dual* is $\hat{G} := \mathrm{Hom}(G, S^1)$. For each $f \in L^2(G)$ fixed we have the Fourier transform $\hat{f} \in L(\hat{G})$ defined by

$$\chi \longmapsto \hat{f}(\chi) := \int_G \bar{\chi}(t) f(t)\, d\mu(t)$$

Here μ is the Haar measure on G. We thus have a map

$$\begin{aligned} L^2(G) &\longrightarrow L^2(\hat{G}) \\ f &\longmapsto \hat{f} \end{aligned}$$

For $G = \mathbb{R}$ we have the group isomorphism

$$\begin{aligned} \mathbb{R} &\xrightarrow{\sim} \hat{\mathbb{R}} \\ s &\longmapsto \left(t \longmapsto e^{2\pi i s t} \right) \end{aligned}$$

So we may think of $\hat{f}$ as $\hat{f} : \mathbb{R} \longrightarrow \mathbb{C}$ as above.

<u>Defn</u> We say that a smooth function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is a Schwartz function if $|f(t)| \ll |t|^{-N}$ as $t \longmapsto \infty$, given any N.

We shall denote the set of these functions by $\mathcal{S}(\mathbb{R})$.

<u>Lemma</u> The map $(*)$ restricts to a map $\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ and $\forall\, f, g \in \mathcal{S}(\mathbb{R})$ we have

$$\widehat{f(\lambda t)}\,(s) = \frac{1}{\lambda}\, \hat{f}\left(\frac{s}{\lambda}\right)$$

Proof

$$(u = t\lambda)$$

$$\widehat{f(\lambda t)}\,(s) = \int_{\mathbb{R}} e^{-2\pi i s t} f(t\lambda)\, dt = \int_{\mathbb{R}} e^{-2\pi i s u \frac{1}{\lambda}} f(u)\, \frac{du}{\lambda}$$

$$= \frac{1}{\lambda}\, \hat{f}\left(\frac{s}{\lambda}\right) \qquad \square$$

Thm (Poisson summation formula) For each $f \in \mathcal{S}(\mathbb{R})$

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

Proof

Let $F(x) := \sum_{n \in \mathbb{Z}} f(x+n)$, periodic of period one. So

$$F(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x} \quad \text{(Fourier series)}$$

where

$$a_n = \int_0^1 F(x) e^{-2\pi i n x} dx = \int_0^1 \sum_{m \in \mathbb{Z}} f(x+m) e^{-2\pi i n x} dx$$

$$= \sum_{m \in \mathbb{Z}} \int_0^1 f(x+m) e^{-2\pi i n x} dx = \sum_{m \in \mathbb{Z}} \int_0^1 f(x+m) e^{-2\pi i (x+m)} d(x+m)$$

$$= \hat{f}(n).$$

Hence

$$\sum_{n \in \mathbb{Z}} f(x+n) = F(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

By evaluating the above at $x = 0$ the result follows $\square$

<u>Propn</u> Let $g(t) := e^{-\pi t^2}$. Then $\hat{g} = g$.

Proof

$$\hat{g}(s) = \int_{\mathbb{R}} e^{-2\pi i x s} e^{-\pi x^2} dx = \int_{\mathbb{R}} e^{-\pi(x^2 + 2ixs)} dx$$

$$= \int_{\mathbb{R}} e^{-\pi(x+is)^2 - \pi s^2} dx = e^{-\pi s^2} \int_{\mathbb{R}} e^{-\pi(x+is)^2} dx$$

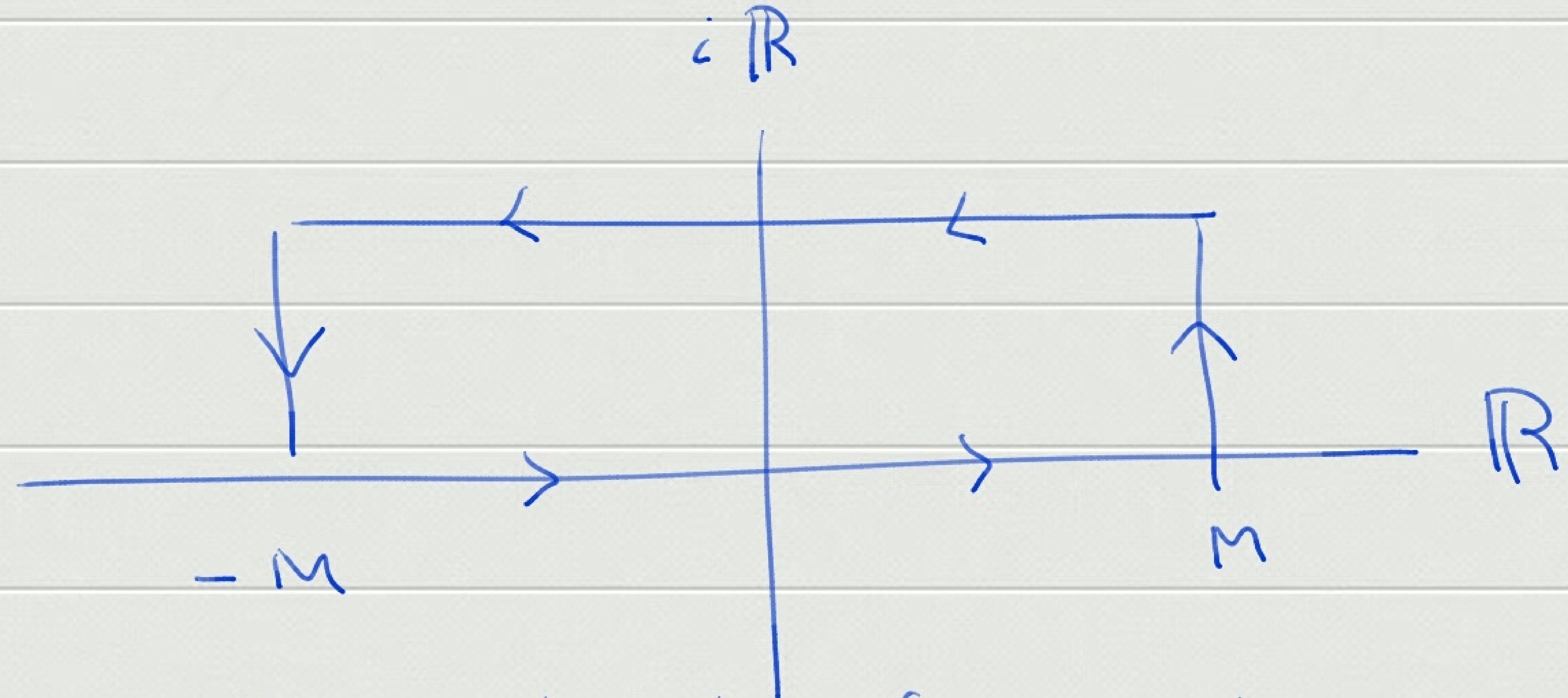
$$= e^{-\pi s^2} \int_{is+\mathbb{R}} e^{-\pi z^2} dz$$

Claim We have

$$\int_{is+\mathbb{R}} e^{-\pi z^2}\, dz \overset{\star}{=} \int_{\mathbb{R}} e^{-\pi x^2}\, dx \overset{\star'}{=} 1$$

Proof of claim

Let's integrate $\omega = e^{-\pi z^2} dz$ along the path



and let $M \to \infty$; the contribution from vertical paths $\to 0$.

The $(\star)$ follows from Cauchy–Goursat. The equality $(\star')$

follows from

$$\int_{\mathbb{R}} e^{-\pi x^2}\,dx = 2\int_0^\infty e^{-\pi x^2}\,dx$$

$$= 2\sqrt{\int_0^\infty e^{-\pi x^2}\,dx \cdot \int_0^\infty e^{-\pi x^2}\,dx}$$

$$= 2\sqrt{\int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-\pi r^2}\, r\, d\theta\, dr}$$

$$= 2\sqrt{\frac{\pi}{2}\left[-\frac{1}{2\pi} e^{-\pi r^2}\right]_0^\infty}$$

$$= 2\sqrt{\frac{\pi}{2}\cdot\frac{1}{2\pi}} = 1 \qquad \square$$

<u>Corollary</u> For each $t \in \mathbb{R}$ let $f_t(x) := e^{-\pi x^2 t}$. Then

$$\hat{f}_t(s) = \frac{1}{\sqrt{t}} e^{-\frac{\pi s^2}{t}}$$

Proof

Clearly $f_t(x) = f_1(\sqrt{t} \cdot x)$. By the lemma we have

$$\hat{f}_t(s) = \widehat{f_1(\sqrt{t} \cdot x)}(s) = \frac{1}{\sqrt{t}} \hat{f}_1\left(\frac{s}{\sqrt{t}}\right)$$

But $f_1(x) = e^{-\pi x^2}$, so the above Prop'n yields

$$\hat{f}_1(s) = f_1(s) = e^{-\pi s^2} \qquad \square$$

Proof of the main identity

Let $f_t(x) = e^{-\pi x^2 t}$, as above, and apply the Poisson summation formula,

$$\theta(t) = \sum_{n \in \mathbb{Z}} f_t(n) = \sum_{n \in \mathbb{Z}} \hat{f}_t(n) = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$$

$$= \frac{1}{\sqrt{t}}\, \theta\left(\frac{1}{t}\right) \qquad \square$$