The Prime Number Theorem.

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

First proofs:

The prime number theorem (PNT) was established in 1896 by Jacques Hadamard and by Charles-Jean de la Vallée Poussin.

→ Hadamard (Versailles, France 1865 – 1963).
→ De la Vallée Poussin (Louvain, Belgium 1866–1962).

Today's proof: (Don Zagier's article, 1997).

→ Don Zagier (Heidelberg, West Germany 1951 —).

Theorem (Prime number theorem).

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

Proof: The proof is by a series of 6 steps.
Specifically, we prove a sequnce of properties of the three functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Phi(s) = \sum_{p} \frac{\log p}{p^s}, \quad \vartheta(x) = \sum_{p \leq x} \log p.$$

($s \in \mathbb{C}$: $s = \sigma + it$ and $x \in \mathbb{R}$); we always use p to denote a prime number.
$\mathrm{Re}(s) = \sigma$

Claim 1. Assume $\delta > 0$. For $\mathrm{Re}(s) \geq 1+\delta$, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly and is holomorphic in $\mathrm{Re}(s) > 1$.

Proof:
$$\left| \zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s} \right| = \left| \sum_{n=N+1}^{\infty} \frac{1}{n^s} \right| \leq \sum_{n=N+1}^{\infty} \left| \frac{1}{n^s} \right|$$

$$\leq \sum_{n=N+1}^{\infty} \frac{1}{n^{1+\delta}} \quad (\text{since } \mathrm{Re}(s) \geq 1+\delta)$$

$$\leq \int_N^{\infty} \frac{du}{u^{1+\delta}} = \frac{1}{N^{\delta} \cdot \delta}.$$

So, given $\varepsilon > 0$, then $\frac{1}{\delta N^{\delta}} < \varepsilon$, as $N \to \infty$, independent of s.

Thus $\left| \zeta(s) - \sum_{n=1}^{\infty} \frac{1}{n^s} \right| < \varepsilon$. ▲

Since $1, \frac{1}{2^s}, \frac{1}{3^s}, \frac{1}{4^s}, \cdots$ are holomorphic in any closed and bounded subset of $\mathrm{Re}(s)>1$ and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly on such subsets, by the Weierstrass's theorem $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is holomorphic in $\mathrm{Re}(s)>1$. ■

In fact $\zeta(s)$ converges absolutely in $\mathrm{Re}(s)>1$:

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty.$$

* $\zeta(s)$ converges absolutely and uniformly on compact subsets of $\mathrm{Re}(s)>1$ and $\zeta(s)$ is holomorphic in that domain.

We have a similar result for the function $\Phi(s) = \sum_p \frac{\log p}{p^s}$

Claim 2. $\Phi(s)$ converges absolutely and uniformly in compact subsets of $\mathrm{Re}(s)>1$. Thus, $\Phi(s)$ represents a holomorphic function for $\mathrm{Re}(s)>1$.

Proof: Let $\mathrm{Re}(s) \geq s_0$, where $s_0 > 1$, then

$$\sum_p \left| \frac{\log p}{p^s} \right| \leq \sum_{n=1}^{\infty} \frac{\log n}{n^{s_0}}$$

But $\sum_{n=1}^{\infty} \frac{\log n}{n^{s_0}} < \infty$, indeed, let ε and δ positive such that $s_0 = 1+\varepsilon+\delta$, since $\frac{\log n}{n^{\delta}} \to 0$ as $n \to \infty$, then

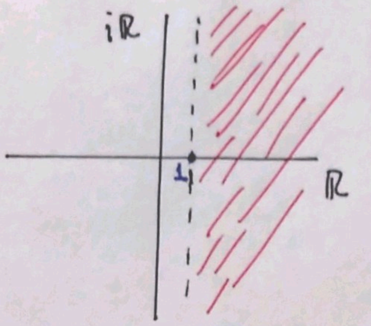
$\exists M$ s.t. $0 \leq \frac{\log n}{n^{\delta}} \leq M \quad \forall n \in \mathbb{N}$.

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{s_0}} = \sum_{n=1}^{\infty} \frac{\log n}{n^{1+\varepsilon+\delta}} = \sum_{n=1}^{\infty} \frac{\log n}{n^{\delta}} \cdot \frac{1}{n^{1+\varepsilon}} \le \sum_{n=1}^{\infty} \frac{M}{n^{1+\varepsilon}} < \infty.$ Result *

Hence $\displaystyle\sum_{p} \left| \frac{\log n}{p^{s}} \right| \le \sum_{n=1}^{\infty} \frac{\log n}{n^{s_0}} < \infty.$ $\left(\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{\sigma}} < \infty \text{ if } \sigma > 1 \right).$

So $\Phi(s)$ converges uniformly for $Re(s) \ge s_0$ $\forall s_0 > 1$.

As in claim 1, by the Weierstrass's theorem $\Phi(s)$ is holomorphic in $Re(s) > 1$. ■



$s = \sigma + it$

$Re(s) > 1$

Part I. $\displaystyle\zeta(s) = \prod_{p} \frac{1}{(1-p^{-s})}$ for $Re(s) > 1$. $(\zeta(s) \neq 0)$.

Proof: From unique factorization and absolute convergence of $\zeta(s)$ we have

$$\zeta(s) = \sum_{r_2, r_3, \dots \ge 0} \frac{1}{(2^{r_2} 3^{r_3} \cdots)^s} = \prod_{p} \left(\sum_{r \ge 0} \frac{1}{p^{rs}} \right) = \prod_{p} \frac{1}{1-p^{-s}}$$ ■

$\rightarrow \zeta(s) \neq 0$: $\displaystyle\left| \frac{1}{\zeta(s)} \right| = \prod_{p} \left| 1 - \frac{1}{p^{s}} \right| \le \prod_{p} \left(1 + \frac{1}{p^{\sigma}} \right)$

$$\le \prod_{p} \left(1 + \frac{1}{p^{\sigma}} + \frac{1}{p^{2\sigma}} + \cdots \right) = \zeta(\sigma) < \infty.$$

Thus $\zeta(s) \neq 0$ for $Re(s) = \sigma > 1$.

Part II.

$\zeta(s) - \frac{1}{s-1}$ extends holomorphically to $\sigma = Re(s) > 0$.

Proof. As before, it is sufficient if we prove uniform convergence on compact subsets of $\sigma > 0$.

Let $K \subset \sigma > 0$. Let $\sigma_0 = \min\{\sigma : \sigma + it \in K\}$. Let $M = \max\{|s| : s \in K\}$

If $\sigma > 1$, then notice that

$$\int_1^\infty \frac{1}{x^s} dx = \frac{x^{-s+1}}{-s+1}\Big|_1^\infty = \frac{1}{1-s}.$$

Thus

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^\infty \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s} dx = \sum_{n=1}^\infty \frac{1}{n^s} - \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{x^s} dx$$

$$= \sum_{n=1}^\infty \frac{1}{n^s} \int_n^{n+1} dx - \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{x^s} dx$$

$$= \sum_{n=1}^\infty \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx.$$

The last series converges absolutely for $\sigma \geq \sigma_0$. Indeed,

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx \right| = \left| -\int_n^{n+1} \int_n^x \frac{du}{u^{s+1}} dx \right|, \text{(since}$$

$$\frac{1}{n^s} - \frac{1}{x^s} = -s \int_n^x \frac{du}{u^{s+1}} = -s \cdot \frac{1}{s u^s}\Big|_n^x = -\frac{1}{x^s} + \frac{1}{n^s} \checkmark)$$

$$\leq |s| \int_n^{n+1} \int_n^x \left| \frac{du}{u^{s+1}} \right| dx$$

$$= |s| \int_n^{n+1} \int_n^x \frac{du}{u^{\sigma+1}} dx$$

Since $\frac{1}{u^{\sigma+1}} \leq \frac{1}{n^{\sigma_0+1}}$ for each $n \leq u \leq x \leq n+1$, then

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| \leq \frac{|s|}{n^{\sigma_0+1}} \int_n^{n+1} \int_n^{n+1} du\, dx = \frac{|s|}{n^{\sigma_0+1}}.$$

Thus, if $s \in K$ then $|s| \leq M$ and then

$$\sum_{n=1}^{\infty} \left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) \right| \leq \sum_{n=1}^{\infty} \frac{M}{n^{\sigma_0+1}} < \infty.$$

And therefore the convergence is uniform in K.
By Weierstrass's theorem, holomorphic for $\sigma > 0$. $\blacksquare$

Part III.

$$\vartheta(x) = O(x).$$

Note: We say that $f(n) = O(g(n))$, if $\exists A > 0$ such that $|f(n)| \leq A g(n)$. $(g(n) > 0 \ \forall n \in \mathbb{R})$.

Since $\vartheta(x)$ is non-decreasing, we need to show that $\exists A > 0$ such that $\vartheta(x) \leq Ax$, for each $x \geq 1$.

By the binomial theorem we have

$$2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} \geq \binom{2n}{n} \geq \prod_{n<p\leq 2n} p$$

The last inequality is true, since

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{(n+1)(n+2)\cdots(2n)}{n!} \geq \prod_{n<p\leq 2n} p$$

since if $n<p\leq 2n$ then it appears in the decomposition of the integer $(n)(n+1)\cdots(2n)/n!$ at least once.

Thus

$$2^{2n} \geq \prod_{n<p\leq 2n} p = e^{\log \prod_{n<p\leq 2n} p} = e^{\sum_{n<p\leq 2n} \log p} = e^{(\vartheta(2n)-\vartheta(n))}.$$

Since log is increasing, then

$$\log 2^{2n} \geq \log\left(e^{(\vartheta(2n)-\vartheta(n))}\right) \Rightarrow \vartheta(2n)-\vartheta(n) \leq 2n\log 2.$$

for all integers $n \geq 1$. For any integer $m \geq 1$ we have

$$\vartheta(2^m) = \sum_{n=1}^{m} (\vartheta(2^n) - \vartheta(2^{m-1})) \leq \sum_{n=1}^{m} 2^n \log 2$$

$$= 2(2^m - 1)\log 2 \leq 2^{m+1}\log 2$$

For any real $x \geq 1$ we can choose an integer $m \geq 1$ so that $2^{m-1} \leq x < 2^m$, and then

$$\vartheta(x) \leq \vartheta(2^m) \leq 2^{m+1}\log 2 = (4\log 2)2^{m-1} \leq (4\log 2)x.$$

Take $A = 4\log 2$. Thus $\vartheta(x) \leq Ax$. ∎

SUMMARY OF $\zeta(s)$:

→ $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ for $Re(s) > 1 \Rightarrow \zeta(s) \neq 0$ for $Re(s) > 1$.

→ Analytic continuation: $\zeta(s) - \frac{1}{s-1} =: \phi(s)$.

$\phi(s)$ is holomorphic on $Re(s) > 0$. Thus $\zeta(s)$ extends to a meromorphic function on $Re(s) > 0$ that has a simple pole at $s = 1$ with residue 1 and no other poles.

We now wish to show that $\zeta(s)$ has no zeros on $Re(s)=1$, this is the key to proving the PNT.
For this we rely on the following lemma.

Lemma. 3.
For all $\sigma, t \in \mathbb{R}$ with $\sigma > 1$ we have

$$|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)| \geq 1$$

Proof: From the relation $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$, we see that for $Re(s) > 1$ we have

$$\log|\zeta(s)| = \log\left|\prod_p \frac{1}{1-p^{-s}}\right| = \sum_p \log\left|\frac{1}{1-p^{-s}}\right| = -\sum_p \log|1-p^{-s}|$$

$$= -\sum_p Re(\log(1-p^{-s})) = \sum_p \sum_{n=1}^{\infty} \frac{Re(p^{-ns})}{n}$$

where we have used the general facts:

→ $\log|z| = Re \log z$

→ $\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ for $|z| < 1$

Applying this to $s = \sigma + it$ yields

$$Re(p^{-ns}) = Re(p^{-n\sigma - int}) = Re(e^{(-n\sigma - int)\log p})$$

$$= Re(e^{-n\sigma \log p}(\cos(nt\log p) - i\sin(nt\log p)))$$

$$\Rightarrow Re(p^{-ns}) = e^{-n\sigma\log p}\cos(nt\log p)$$

$$= \frac{\cos(nt\log p)}{p^{n\sigma}}$$

$$\Rightarrow \sum_p \sum_{n=1}^{\infty} \frac{Re(p^{-ns})}{n} = \sum_p \sum_{n=1}^{\infty} \frac{\cos(nt\log p)}{n p^{n\sigma}}$$

Thus $\log|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)|$

$$= \sum_p \sum_{n=1}^{\infty} \frac{3+4\cos(nt\log p)+\cos(2nt\log p)}{np^{n\sigma}}$$

From the identity $\cos(2\theta) = 2\cos^2\theta - 1$ implies

$$3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geq 0$$

Taking $\theta = nt\log p$ yields $\log|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| \geq 0$

$\blacksquare$

Corollary 4.

$\zeta(s)$ has no zeros on $\mathrm{Re}(s) \geq 1$.

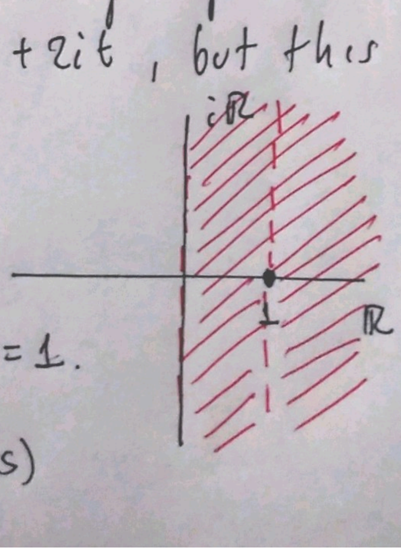
Proof: From part I we know that $\zeta(s)$ has no zeros on $\mathrm{Re}(s) > 1$, so suppose $\zeta(1+it) = 0$ for some $t \in \mathbb{R}$. Then $t \neq 0$, since $\zeta(s)$ has a pole at $s=1$, and we know that $\zeta(s)$ does not have a pole at $1+2it \neq 1$, by part II. We therefore must have

$$\lim_{x \to 1} |\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)| = 0,$$

since $\zeta(s)$ has a simple pole at $s=1$, a zero at $1+it$, and no pole at $1+2it$, but this contradicts lemma 3. $\blacksquare$

$\blacksquare$

$\phi(s) = \zeta(s) - \frac{1}{s-1}$



$\zeta(s)$ has no zeros on $\mathrm{Re}(s) \geq 1$

simple pole at $s=1$.

$\zeta(s) = \frac{1}{s-1} + \phi(s)$

Part IV.

$\zeta(s) \neq 0$ and $\Phi(s) - \frac{1}{s-1}$ is holomorphic for $Re(s) \geq 1$

Proof:

That $\zeta(s) \neq 0$ for $Re(s) \geq 1$ is exactly Corollary 4. $\blacksquare$

$\rightarrow$ The logarithmic derivative $\frac{\zeta'(s)}{\zeta(s)}$ of $\zeta(s)$ is meromorphic on $Re(s) > 0$, since (the extension of) $\zeta(s)$ is.

In term of the Euler product we have

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \Rightarrow \log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}}$$

$$\Leftrightarrow \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = -\sum_p \log(1-p^{-s}).$$

That is $\log \zeta(s) = -\sum_p \log(1-p^{-s})$. Then, taking $\frac{d}{ds}$ both sides:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\frac{d}{ds}(1-p^{-s})}{1-p^{-s}}, \text{ but } \frac{d}{ds}(1-p^{-s}) = p^{-s}\log p.$$

Then $-\sum_p \frac{\frac{d}{ds}(1-p^{-s})}{1-p^{-s}} = -\sum_p \frac{p^{-s}\log p}{1-p^{-s}}$. Then

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{p^{-s}\log p}{1-p^{-s}}$$

$$= \sum_p \frac{\log p}{p^s - 1}$$

$$= \sum_p \frac{p^s \log p}{p^s(p^s-1)}$$

$$= \sum_p \frac{p^s \log p - \log p + \log p}{p^s(p^s-1)}$$

$$= \sum_p \frac{(p^s-1)\log p + \log p}{p^s(p^s-1)}$$

$$= \sum_p \frac{\log p}{p^s} + \sum_p \frac{\log p}{p^s(p^s-1)}$$

Therefore

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s} + \sum_p \frac{\log p}{p^s(p^s-1)}$$

$$= \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}.$$

The final series converges for $Re(s) > 1/2$. Indeed,

$$|p^s-1| \geq |p^s| - 1 = p^\sigma - 1 > \frac{p^\sigma}{10} \Rightarrow \frac{1}{|p^s-1|} < \frac{10}{p^\sigma}$$

$$\Rightarrow \sum_p \left| \frac{\log p}{p^s(p^s-1)} \right| = \sum_p \frac{\log p}{|p^s||p^s-1|} \leq 10 \sum_p \frac{\log p}{p^{2\sigma}} < \infty \text{ by } *$$

So it is absolutely convergent.

Let $F(s) = \sum_p \frac{\log p}{p^s(p^s-1)} \Rightarrow |F(s)| \leq 10 \sum_p \frac{\log p}{p^{2\sigma}} \leq 10 \sum_{n=1}^{\infty} \frac{\log n}{n^{2\sigma}}$

$$\Rightarrow \left|F(s) - \sum_{n=1}^{N} \frac{10 \log n}{n^{2\sigma}}\right| \leq \left|10 \sum_{n=N+1}^{\infty} \frac{\log n}{n^{2\sigma}}\right| \leq \sum_{n=N+1}^{\infty} \frac{10 \log n}{n^{2\sigma}} < 10 \int_N^\infty \frac{\log u}{u^{2\sigma}}\, du < \infty.$$

Thus $F(s) = \sum_p \frac{\log p}{p^s(p^s-1)}$ converges uniformly on $Re(s) > \frac{1}{2}$.
By Weierstrass's theorem $F(s)$ represents a holomorphic function for $Re(s) > \frac{1}{2}$.

$-\frac{\zeta'(s)}{\zeta(s)}$ is holomorphic on $Re(s) > 1$, since $\zeta(s)$ has no zeros or poles in this region; moreover $-\frac{\zeta'(s)}{\zeta(s)}$ has only a simple pole of residue 1 at $s=1$ on $Re(s)=1$, since $\zeta(s)$ has no zeros on $Re(s)=1$. It follows that $\Phi(s) - \frac{1}{s-1}$ extends to a meromorphic function on $Re(s) > \frac{1}{2}$ that is holomorphic on $Re(s) \geq 1$. ∎

The Laplace transform

Definition 5. Let $h: \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a piecewise continuous function. The *Laplace transform* of h is the complex function defined by

$$(\mathcal{L}h)(s) := \int_0^\infty e^{-st} h(t)\,dt;$$

it is a holomorphic function on $Re(s) > c$ for any $c \in \mathbb{R}$ for which $h(t) = O(e^{ct})$.

Properties:

- $\mathcal{L}(g+h) = \mathcal{L}g + \mathcal{L}h$
- $\mathcal{L}(ah) = a\mathcal{L}h, \ \forall a \in \mathbb{R}$
- If $h(t) = a \in \mathbb{R}$ is constant then $\mathcal{L}h = \frac{a}{s}$
- $\mathcal{L}(e^{at}h)(s) = (\mathcal{L}h)(s-a) \ \forall a \in \mathbb{R}$.

Teorem 6 (Analytic Theorem).
Let $f(t)$ $(t \geq 0)$ be a bounded and locally integrable function and suppose that the function

$$g(z) = \int_0^\infty f(t) e^{-zt} dt = (\mathcal{L} f)(z) \quad , \quad (Re(z) > 0)$$

extends holomorphically to $R(z) \geq 0$. Then $\int_0^\infty f(t) dt$ exists and $g(0) = \int_0^\infty f(t) dt$.

Part V. $\int_1^\infty \frac{\vartheta(x) - x}{x^2} dx$ is a convergent integral.

Proof: For $Re(s) > 1$ we have

$$\Phi(s) = \sum_p \frac{\log p}{p^s} = \int_1^\infty \frac{d\vartheta(x)}{x^s} = s \int_1^\infty \frac{\vartheta(x)}{x^{s+1}} dx = s \int_0^\infty e^{-st} \vartheta(e^t) dt$$

$$\Rightarrow \quad \Phi(s) = s \int_0^\infty \vartheta(e^t) e^{-st} dt$$

$$\Rightarrow \quad (\mathcal{L} \vartheta(e^t))(s) = \frac{\Phi(s)}{s}.$$

Notice that

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2} dx = \int_0^\infty \frac{\vartheta(e^t) - e^t}{e^{2t}} \cdot e^t dt = \int_0^\infty \frac{\vartheta(e^t) - e^t}{e^t} dt$$

$$= \int_0^\infty (\vartheta(e^t) e^{-t} - 1) dt$$

So, the Laplace transform of $\vartheta(e^t)e^{-t}-1$ is

$$\mathcal{L}\{\vartheta(e^t)e^{-t}-1\}(s) = \mathcal{L}\{\vartheta(e^t)e^{-t}\}(s) - \mathcal{L}\{1\}(s)$$
$$= \left(\mathcal{L}\{\vartheta(e^t)\}\right)(s+1) - \frac{1}{s}$$
$$= \frac{\Phi(s+1)}{s+1} - \frac{1}{s} \quad \text{for } \operatorname{Re}(s)>1.$$

We can extend the function $\Phi(s)-\frac{1}{s}$ and the function $\frac{\Phi(s+1)}{s+1}-\frac{1}{s}$ to meromorphic functions on $\operatorname{Re}(s)>-\frac{1}{2}$ that are holomorphic on $\operatorname{Re}(s)\geq 0$. Indeed, remember that $\Phi(s)-\frac{1}{s-1}$ is meromorphic on $\operatorname{Re}(s)>\frac{1}{2}$ and holomorphic on $\operatorname{Re}(s)\geq 1$, so let $s\mapsto s+1$, then $\Phi(s+1)-\frac{1}{s}$ is meromorphic on $\operatorname{Re}(s)>-\frac{1}{2}$ and holomorphic on $\operatorname{Re}(s)\geq 0$. Similarly $\frac{\Phi(s+1)}{s+1}-\frac{1}{s}$ is meromorphic on $\operatorname{Re}(s)>-\frac{1}{2}$, and notice that

$$\frac{\Phi(s+1)}{s+1}-\frac{1}{s} = \frac{1}{s+1}\left(\Phi(s+1)-\frac{1}{s}\right)-\frac{1}{s+1} \quad \text{is holomorphic}$$

on $\operatorname{Re}(s)\geq 0$, since it is a sum of products of holomorphic functions on $\operatorname{Re}(s)\geq 0$.

Therefore the conditions of the Analytical Theorem are fulfilled with

$$f(t)=\vartheta(e^t)e^{-t}-1 \quad \text{and} \quad g(z)=\frac{\Phi(z+1)}{z+1}-\frac{1}{z}.$$

Thus $\int_1^{\infty} \frac{\vartheta(x)-x}{x^2}\,dx$ converges. ∎

Part VI.

$\vartheta(x) \sim x.$

Proof: We want to show that $\lim\limits_{x\to\infty} \frac{\vartheta(x)}{x} = 1.$

By contradiction, that is, there is a $\lambda > 1$ such that $\frac{\vartheta(x)}{x} \geq \lambda$ for x arbitrary large. Since $\vartheta(x)$ is non-decreasing, then $\vartheta(t) \geq \vartheta(x) \geq \lambda x$ for $x \leq t \leq \lambda x$. Thus

$$\int_x^{\lambda x} \frac{\vartheta(t)-t}{t^2}\,dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt.$$

Let $t = xu$ we have

$$\int_x^{\lambda x} \frac{\vartheta(t)-t}{t^2}\,dt \geq \int_1^{\lambda} \frac{\lambda - u}{u^2}\,du > 0 \quad ! \text{ contradicting (V).}$$

Similarly, if $\vartheta(x) \leq \lambda x$ for large values of x and $\lambda < 1$, then

$$\int_{\lambda x}^{x} \frac{\vartheta(t)-t}{t^2}\,dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2}\,dt = \int_{\lambda}^{1} \frac{\lambda - t}{t^2}\,dt < 0 \quad ! \text{ contradicting (V).}$$

The prime number theorem follows easily from (VI), ∎ since

$$\vartheta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x)\log x$$

Also we have that for all $\varepsilon > 0$,

$$\vartheta(x) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \geq \sum_{x^{1-\varepsilon} \leq p \leq x} (1-\varepsilon) \log x$$

$$= (1-\varepsilon) \log x \left[\pi(x) - \pi(x^{1-\varepsilon}) \right].$$

Since $\pi(x^{1-\varepsilon}) \leq x^{1-\varepsilon}$, then

$$\vartheta(x) \geq (1-\varepsilon) \log x \left[\pi(x) - x^{1-\varepsilon} \right]$$

Thus $$\frac{\vartheta(x)}{\log x} \leq \pi(x) \leq \frac{\vartheta(x)}{(1-\varepsilon)\log x} + x^{1-\varepsilon}$$

$$\Rightarrow \frac{\vartheta(x)}{x} \leq \frac{\pi(x) \log x}{x} \leq \frac{\vartheta(x)}{x} \cdot \frac{1}{1-\varepsilon} + \frac{\log x}{x^{\varepsilon}}$$

Therefore

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} \leq \lim_{x \to \infty} \frac{\pi(x) \log x}{x} \leq \lim_{x \to \infty} \frac{\vartheta(x)}{x} \cdot \frac{1}{1-\varepsilon} + \frac{\log x}{x^{\varepsilon}}$$

$$\Rightarrow \quad 1 \leq \lim_{x \to \infty} \frac{\pi(x) \log x}{x} \leq \frac{1}{1-\varepsilon}$$

When $\varepsilon \to 0$, we have

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$$

That is $\pi(x) \sim \dfrac{x}{\log x}$. $\blacksquare$

Theorem. (Weierstrass's Theorem for Series)

Assume $f_1(z), f_2(z), f_3(z), \ldots$

are holomorphic in an open set D, and $\sum_{i=1}^{\infty} f_i(z)$

converges uniformly on every closed and bounded subset of D.

Then

$$F(z) = \sum_{i=1}^{\infty} f_i(z) \text{ is holomorphic on } D.$$