The Analytic Theorem.

Theorem (Analytic Theorem).
Let $f(t)$ $(t \geq 0)$ be a bounded and locally integrable function and suppose that the function

$$g(z) = \int_0^\infty f(t) e^{-zt} dt \quad (Re(z) > 0)$$

extends holomorphically to $Re(z) \geq 0$. Then $\int_0^\infty f(t) dt$ exists

any $g(0) = \int_0^\infty f(t) dt$.

Proof. Let $z = x + iy$.
For $T > 0$, the function $g_T(z) = \int_0^T f(t) e^{-zt} dt$ is entire.
We must show that

$$\lim_{T \to \infty} g_T(0) = g(0).$$

Let R be large and let C be the boundary of the region

$$\{ z \in \mathbb{C} : |z| < R, \ -\theta \leq Re(z) = x \},$$

where θ is a function of R such that $g(z)$ is analytic on C.
In other words: Since $g(z)$ is analytic on $x \geq 0$ then

$g(z)$ is analytic in the line segment that goins $-Ri$ with Ri. For each point z of this segment there is a nbhd V_z where $g(z)$ is analytic. Therefore we have an open cover of the compact set $\{iy : -R \leq y \leq R\}$. Extracting a finite subcover we see that there exists $\theta > 0$ such that $g(z)$ is analytic within and on the contour C.

$\to g(z)$ is analytic on C and inside of C.

We employ Cauchy's Integral Formula to estimate the size of $g(0) - g_T(z)$.

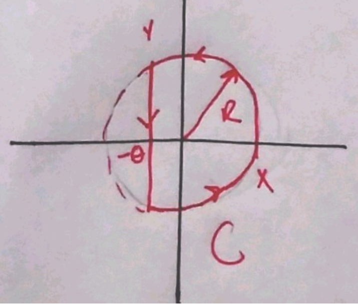
$\to$ Cauchy Integral Formula. If Γ is a simple closed curve and if f is analytic on Γ and inside of Γ, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz.$$

Let $z_0 = 0$, then

$$g(0) - g_T(0) = \frac{1}{2\pi i} \oint_C \frac{g(z)}{z} \, dz - \frac{1}{2\pi i} \oint_C \frac{g_T(z)}{z} \, dz$$

$$= \frac{1}{2\pi i} \oint_C (g(z) - g_T(z)) \frac{1}{z} \, dz.$$



Observe that we may modify the lost equation by replacing g, g_T with their products with e^{zT} without changing the value of the integral because $g(0)e^{0\cdot T} = g(0)$ and $g_T(0)e^{0\cdot T} = g_T(0)$. Futhermore, we can add

$$\frac{1}{R^2}\cdot\frac{1}{2\pi i}\oint_C (g(z)-g_T(z))\,z\,dz$$

to the right hand side because $\frac{z}{R^2}(g(z)-g_T(z))$ is analytic over C and inside of C, then by Cauchy's Theorem

$$\frac{1}{R^2}\cdot\frac{1}{2\pi i}\oint_C (g(z)-g_T(z))\,z\,dz = 0.$$

Thus we have

$$g(0)-g_T(0) = \frac{1}{2\pi i}\oint_C (g(z)-g_T(z))\frac{1}{z}\,dz$$

$$= \frac{1}{2\pi i}\oint_C (g(z)-g_T(z))\frac{e^{zT}}{z}\,dz + \frac{1}{R^2}\cdot\frac{1}{2\pi i}\oint_C (g(z)-g_T(z))\,z\,dz$$

$$= \frac{1}{2\pi i}\oint_C (g(z)-g_T(z))\frac{e^{zT}}{z}\,dz + \frac{1}{R^2}\cdot\frac{1}{2\pi i}\oint_C (g(z)-g_T(z))e^{zT}\cdot z\,dz$$

$$= \frac{1}{2\pi i}\oint_C (g(z)-g_T(z))\,e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz.$$

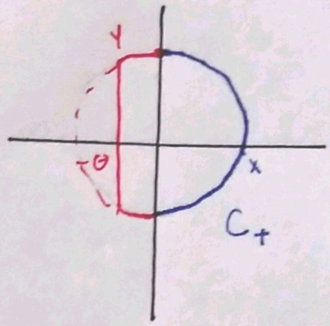
$$= \frac{1}{2\pi i}\oint_C (g(z)-g_T(z))e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\,dz.$$

Define $C_+ = C \cap \{x>0\}$. Let $B = \max\{|f(t)| : t \geq 0\}$. Then

$$|g(z) - g_T(z)| = \left| \int_0^\infty f(t) e^{-zt} dt - \int_T^\infty f(t) e^{-zt} dt \right|$$

$$= \left| \int_T^\infty f(t) e^{-zt} dt \right|$$

$$\leq \int_T^\infty |f(t)| \, |e^{-zt}| \, dt$$



$$\leq B \int_T^\infty e^{-xt} dt$$

$$= B \frac{e^{-Tx}}{x} \quad \text{(since } x>0\text{)}.$$

On the other hand, $z = x+iy = Re^{i\theta}$ with $\theta \in [0, 2\pi]$

$$\left| e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z} \right| = |e^{zT}| \left| \left(1+\frac{z^2}{R^2}\right)\frac{1}{z} \right| = e^{xT} \left| \frac{1}{z} + \frac{z}{R^2} \right|$$

but, $R\cos\theta = x \Rightarrow \cos\theta = \frac{x}{R}$, then

$$\left| \frac{1}{z} + \frac{z}{R^2} \right| = \left| \frac{R}{z} + \frac{z}{R} \right| \frac{1}{R} = \left| e^{-i\theta} + e^{i\theta} \right| \cdot \frac{1}{R}$$

$$= |2 \cdot \cos\theta| \cdot \frac{1}{R} = 2 \cdot \frac{x}{R} \cdot \frac{1}{R} = \frac{2x}{R^2}.$$

Thus $$\left| e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z} \right| = e^{xT} \cdot \frac{2x}{R^2}.$$

And therefore

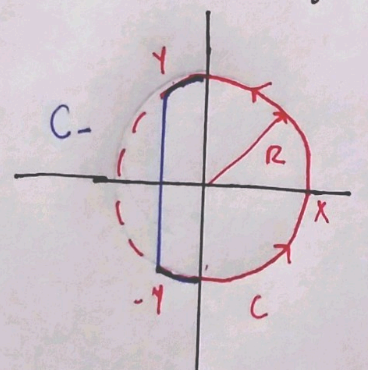
$$|g(0)-g_T(0)| = \left| \frac{1}{2\pi i} \int_{C^+} (g(z)-g_T(z)) e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\, dz \right|$$

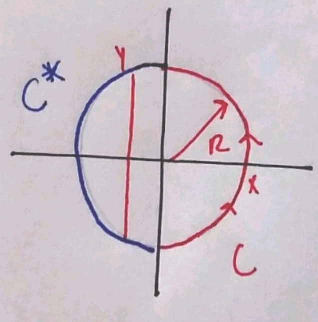
$$\leq \frac{1}{2\pi} \cdot \frac{B e^{-Tx}}{\cancel{x}} \cdot \frac{e^{xT}\cdot 2\cancel{x}}{R^2} \cdot \pi R$$

$$= \frac{B}{R}.$$

Define $C_- = C \cap \{z = x+iy : x<0\}$. We will take $g(z)$ and $g_T(z)$ separately.

Since $g_T(z)$ is entire, then the integration contour C_- for the integral involving can be replaced by the semicircle $C^* = \{z = x+iy : |z| \le R,\ x<0\}$





Then, we have to analyze the following integrals:

$$I_1(T,R) = \frac{1}{2\pi i} \int_{C^*} g_T(z) e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\, dz,$$

$$I_2(T,R) = \frac{1}{2\pi i} \int_{C_-} g(z)\, e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}\, dz$$

→ For I_1:

Notice that

$$|g_T(z)| = \left| \int_0^T f(t) e^{-zt} dt \right|$$
$$\leq \int_0^T |f(t)| \, |e^{-zt}| dt$$
$$\leq B \int_{-\infty}^T e^{-xt} dt \quad (\text{since } x<0)$$
$$= B \frac{e^{-xT}}{|x|}.$$

For $z \in C^*$ we have

$$\left| e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z} \right| = \frac{2|x|}{R^2} e^{xT}, \text{ where } x<0,$$

then $$|I_1(T,R)| \leq \frac{1}{2\pi} \int_{C^*} |g_T(z)| \left| e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{1}{z} \right| dz$$
$$\leq \frac{1}{2\pi} \cdot B\frac{e^{-xT}}{|x|} \cdot \frac{2|x| e^{xT}}{R^2} \cdot \pi R$$
$$= \frac{B}{R}.$$

→ For I_2:

We have that this integral tends to 0 as $T \to \infty$ because the integrand is the product of the function $g(z)\left(1+\frac{z^2}{R^2}\right)\frac{1}{z}$, which is independent of T, and the function e^{zT}, which goes to 0 rapidly and uniformly on compact sets as $T \to \infty$ in the half-plane $x < 0$, that is

$$\lim_{T\to\infty} |I_2(T,R)| = 0$$

$$|g(0) - g_T(0)| = \left| \frac{1}{2\pi i} \oint_C (g(z) - g_T(z))\, e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\, dz \right|$$

$$\leq \frac{1}{2\pi} \int_{C_+} \left|(g(z) - g_T(z))\, e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| dz$$

$$+ \frac{1}{2\pi} \int_{C_-} \left|(g(z) - g_T(z))\, e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| dz$$

$$\leq \frac{B}{R} + \frac{B}{R} + |I_2(T,R)|$$

So

$$\limsup_{T\to\infty} |g(0) - g_T(0)| \leq \limsup_{T\to\infty} \left(\frac{2B}{R} + |I_2(T,R)|\right)$$

$$= \frac{2B}{R}.$$

Since R was arbitrary, then $\lim_{T\to\infty} g_T(0) = g(0)$.

$\blacksquare$