Appendix 1 Further remarks on Dirichlet series

Prop^n Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ be Dirichlet series. We have $F(s)\,G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$,

where

$$h(n) = \sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right)$$

Proof

[Ex.]

Moreover, we have $\forall\, r \in \{1, 2, 3, 4, \dots\}$:

$$\zeta(s)\,\zeta(s-r) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s}$$

where $\sigma_r(n) := \sum_{d \mid n} d^r$. This Dirichlet series corresponds in a natural way to a modular form .

$$E_k : \mathcal{H} \longrightarrow \mathbb{C}$$

where $k \in \{4, 6, 8, \dots\}$ with Fourier expansion

$$E_k(\tau) = 1 - 2\,\frac{k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n \qquad \left(q = e^{2\pi i \tau}\right).$$

Here $\mathcal{H}$ denotes the Poincaré upper half plane

$$\mathcal{H} := \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}.$$

Another modular form is

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24 q^2 + 252 q^3 - 1472 q^4 + \ldots$$

and we'll show the identity

$$\Delta = \frac{E_4^3 - E_6^2}{1728},$$

which has a nice interpretation in the theory of elliptic curves.

We have

$$\frac{\Delta'(\tau)}{\Delta(\tau)} = 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} n \frac{q^n}{1-q^n} \right)$$

$$= 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n \, q^{nm} \right)$$

$$= 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right).$$

Therefore

$$\frac{d}{d\tau} \log \Delta(\tau) = 2\pi i \, E_2(\tau)$$

which is <u>not</u> a modular form — it is a quasimodular form.

Quoting Wiener, the prime numbers bear an exceedingly close relation to the series of the form

$$\sum_n a_n \frac{x^n}{1 - x^n}$$

known as Lambert series.[1] He puts $a_n = \Lambda(n)$, the Mangoldt function. Then he applies his Tauberian theorem to get a beautiful proof of the PNT.

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1 Ch. IV of Wiener, N., *Tauberian Theorems*, The Annals of Mathematics, 33, No. 1, 1932, pp. 1–100.

Following Ex 4, Ch XIII of Apostol's book, let

$$M(x) = \sum_{n \leq x} \mu(n)$$

It turns out that the PNT is equivalent to $\forall\, \varepsilon > 0 \;\; \exists\, N_\varepsilon$ s.t. $\forall\, x > N_\varepsilon$:

$$\frac{1}{x} |M(x)| \leq \varepsilon . \qquad (*)$$

The exercise shows how the Riemann hypothesis (i.e. all non trivial zeroes s of $\zeta(s)$ have $\mathrm{Re}(s) = \frac{1}{2}$) is equivalent to replacing $(*)$ by the stronger condition

$$\frac{1}{x} |M(x)| \leq x^{-\frac{1}{2}+\varepsilon} .$$

Indeed, first use the property

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

to show that

$$\zeta(s) \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right) = 1,$$

thus

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

But we may view the right hand side of the latter eq'n as a Stieltjes integral

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \int_0^\infty x^{-s}\, dM(x).$$

The integration by parts formula yields

$$\frac{1}{s\,\zeta(s)} = \int_0^\infty x^{-s} M(x)\,\frac{dx}{x} = (\mathcal{M}\, M)(-s).$$

But $\mathcal{M}\, M$ is the Mellin transform of M, so Mellin's inversion formula implies that

$$M(x) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{x^s}{s\,\zeta(s)}\, ds \qquad (\sigma = \mathrm{Re}(s)) \qquad (\star)$$

$\forall\ \sigma \in (1, 2)$. But RH makes $(\star)$ convergent for $\sigma \in (1/2, 2)$.

Therefore $\forall\ \alpha \in (1/2, \infty)$:

$$M(x) = O(x^\alpha)$$

as $x \longrightarrow \infty$.