Lecture 8 Theta series

Given any positive definite integer-valued quadratic form

$$Q(x_1, \ldots, x_m)$$

we may define the the theta series of Q as

$$\theta_Q(\tau) := \sum_{n=0}^{\infty} r_Q(n)\, q^n,$$

where

$$r_Q(n) = \#\{(x_1, \ldots, x_m) \in \mathbb{Z}^m \mid Q(x_1, \ldots, x_m) = n\},$$

as before $q = e^{2\pi i \tau}$ and

$$\tau \in \mathcal{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}.$$

For example, if $n = 1$ and $Q(x) = x^2$ then we get Jacobi's theta function

$$\theta_Q(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

We have

$$\left.\begin{aligned} \theta(\tau+1) &= \theta(\tau) \\ \theta\left(\frac{-1}{4\tau}\right) &= \sqrt{\frac{2\tau}{i}}\, \theta(\tau) \qquad (\text{see Lecture 4}) \end{aligned}\right\} \star$$

The functional eq'ns ($\star$) lead us to discuss the basics of the theory of modular forms.

The modular group $\Gamma := PSL_2(\mathbb{Z})$ acts faithfully on the Poincaré upper half space via Möbius transformations

$$\begin{aligned} \mathcal{H} &\longrightarrow \mathcal{H} \\ \tau &\longmapsto \frac{a\tau + b}{c\tau + d} \end{aligned} \quad , \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma .$$

For each $N \in \{1, 2, 3, \ldots\}$ may define the *principal congruence subgroup* as the kernel of the reduction homomorphism

$$\Gamma(N) := \mathrm{Ker}\left(\Gamma \xrightarrow{\text{red}} \tilde{\Gamma} \right),$$

induced by

$$\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} .$$

A *congruence subgroup* $H \leq \Gamma$ is one that contains $\Gamma(N)$, for some N. A key example is the ~~Hecke~~ congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

We'll focus our discussion on these ones. Consider the action of $\Gamma_0(N)$ on $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ and write

$$Y_0(N) := \Gamma_0(N) \backslash \mathcal{H}$$

$$X_0(N) := \Gamma_0(N) \backslash \mathcal{H}^*$$

It is easy to see that

$$X_0(N) = Y_0(N) \cup \{\text{finite \# points}\}$$

The above finite number of points are known as the cusps.

<u>Ex.</u> Show that if $N = p$ prime, then we have just two cusps, namely, the class of 0 and the class of ∞.

Now we are ready to introduce the following.

<u>Def'n</u> Given $k \in \mathbb{Z}_{\geq 1}$, $N \in \{1, 2, 3, \dots\}$ and a Dirichlet character, $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$, a modular form on $\Gamma_0(N)$ of weight k and character χ is a holomorphic function

$$f : \mathcal{H} \longrightarrow \mathbb{C}$$

s.t.

(1) $$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)\,(c\tau+d)^k f(\tau)$$

(2) $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$, $q = e^{2\pi i \tau}$, and similarly for the other cusps.

A family of examples is given by the Eisenstein series

$$E_{k,\chi}(\tau) := 1 - \frac{4k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n)\, q^n ,$$

where $B_{k,\chi}$ is the k-th Bernoulli number attached to χ,

$$\sum_{a=1}^{N} \chi(a) \frac{t e^{at}}{e^{Nt}-1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}$$

and

$$\sigma_{r,\chi}(n) := \sum_{d \mid n} \chi(d)\, d^r .$$

Ex. Let χ_{-4} denote the Dirichlet character s.t.

$$\chi_{-4}(p) = (-1)^{\frac{p-1}{2}} = \left(\frac{-1}{p}\right)$$

for each odd prime p. Then we have the Eisenstein series on $\Gamma_0(4)$ of weight $k=1$ and character χ_{-4},

$$E_{1,\chi_{-4}}(\tau) = 1 + 4\sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{-4}(d)\right)q^n.$$

Note that

$$\chi_{-4}(n) = \begin{cases} +1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4, \\ 0, & \text{if } 2|n. \end{cases}$$

Note that the square of Jacobi's theta series gives

$$\left(\sum_{n \in \mathbb{Z}} q^{n^2} \right)^2 = \Theta_Q(\tau),$$

where $Q(x_1, x_2) = x_1^2 + x_2^2$ and the functional equations (1) and (2) imply that

$$\Theta_Q \in M_1(\Gamma(4), \chi_{-4}).$$

The first few terms are

$$\Theta_Q(\tau) = 1 + 4q + 4q^2 + 4q^4 + 10q^5 + 4q^8 + \ldots$$

Let $M_k(G)$ denote the $\mathbb{C}$-vector space of modular forms on G.

<u>Thm</u> If G is a discrete subgroup of $SL_2(\mathbb{R})$ s.t.

$$V := \mathrm{vol}(G \backslash \mathcal{H}) < \infty,$$

then

$$\dim(M_k(G)) \leq \frac{kV}{4\pi} + 1.$$

Proof

[We'll give a proof soon.]

We'll show that the above theorem gives

$$\dim\left(M_1\left(\Gamma(4), \chi_{-4}\right)\right) = 1.$$

Therefore $\exists\ \alpha \in \mathbb{C}$ s.t. $\alpha E_{1,\chi_{-4}} = \theta_Q$,

where $Q(x_1, x_2) = x_1^2 + x_2^2$. But

$$E_{1,\chi_{-4}}(\tau) = 1 + \cdots$$

Hence $\alpha = 1$ and we thus get the following.

Thm (Fermat) Every prime number $p \equiv 1 \pmod 4$ is a sum of two squares

Proof

For each prime number p,

$$r_Q(p) = 4\left(1 + (-1)^{\frac{p-1}{2}}\right) = \begin{cases} 8 & \text{if } p \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and $8 \neq 0$ $\square$

Moreover, we have for the 4-th power of Jacobi's theta function

$$\Theta_Q(\tau) = \left(\sum_{n \in \mathbb{Z}} q^{n^2} \right)^4 = 1 + 8q + \cdots$$

where $Q(x_1, \dots, x_4) = x_1^2 + \dots + x_4^2$ and similarly

$$\Theta_Q \in M_2(\Gamma_0(4), \chi),$$

where

$$\chi(n) = \begin{cases} 1, & \text{if } n \not\equiv 0 \pmod{4}, \\ 0, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Similarily $\Theta_Q = E_{2,\chi}$ and we have the following.

Thm (Lagrange) Every prime p may be expressed as

$$p = x_1^2 + \dots + x_4^2, \quad (x_1, \dots, x_4) \in \mathbb{Z}^4.$$

Proof

If $Q(x_1, \dots, x_4) = x_1^2 + \dots + x_4^2$, then $\forall$ prime p

$$r_Q(p) = 8(1 + \chi(p)) = 8(1 + p) > 0 \qquad \square$$