Lecture 9 On Petersson's partition limit formula

Fix a prime number $p \equiv 1 \pmod 4$ and consider the Dirichlet character

$$\chi(\cdot) = \left(\frac{\cdot}{p}\right)$$

The Eisenstein series $E_{2,\chi}(\tau)$ may be obtained as the logarithmic derivative of

$$\breve{u}(\tau) := q^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty} (1-q^n)^{\chi(n)},$$

which is a function on $X_\chi(p) := \Gamma_\chi(p) \backslash \mathcal{H}^*$. The latter is known from the work of Shimura on Real Multiplication. (We'll discuss RM later.)

Here

$$\Gamma_\chi(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p) \;\middle|\; \chi(a) = 1 \right\}.$$

The Fricke involution

$$\begin{aligned} \mathcal{H} &\xrightarrow{w_p} \mathcal{H} \\ \tau &\longmapsto -\frac{1}{p\tau} \end{aligned}$$

is well-defined on $X_\chi(p)$. The composition

$$u = \check{u} \circ w_p$$

is the function of Ogg and Ligozat used in Mazur, *Modular curves and the Eisenstein ideal*, IHES 47, 1977, pp. 33–186.

Remark From the above work of Mazur we know that the only possible torsion subgroups of $E(\mathbb{Q})$, where E is an elliptic curve over $\mathbb{Q}$, are

$$C_1, C_2, \ldots, C_{10}, C_{12}; \quad C_2 \times C_2, \; C_4 \times C_2, \; C_6 \times C_2, \; C_8 \times C_2.$$

The function $u(\tau)$ may be described using Klein's forms as follows. So let's consider the *Weierstraß* σ-function of a lattice $\Lambda \subseteq \mathbb{C}$,

$$z \mapsto \sigma(z, \Lambda) := z \prod_{\omega \in \Lambda - \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

The *Weierstraß* ζ-function is

$$\zeta(z;\Lambda) = \frac{d}{dz} \log \sigma(z;\Lambda).$$

So we have a well-defined quasi period function

$$\Lambda \longrightarrow \mathbb{C}$$

$$\omega \longmapsto \eta(\omega;\Lambda) := \zeta(z+\omega;\Lambda) - \zeta(z;\Lambda)$$

Moreover, $-\zeta'(z;\Lambda) = \wp(z;\Lambda)$, the *Weierstraß* $\wp$-function

$$\wp(z;\Lambda) := \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right].$$

Then the *Klein form* of Λ is

$$\mathfrak{k}(z;\Lambda) := e^{-\frac{1}{2}\eta(z;\Lambda)z}\,\sigma(z;\Lambda),$$

so we may define $\mathfrak{k}_a(\tau) := \mathfrak{k}(z,\Lambda_\tau)$, where the point $a = (a_1, a_2) \in \mathbb{R}^2$ is uniquely determined by

$$z = a_1\tau + a_2,$$

with $\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$, $\tau \in \mathcal{H}$. Following Mazur,

$$u(\tau) := \prod_{r=1}^{\frac{p-1}{2}} \mathfrak{k}_{(0,r/p)}(\tau)^{\chi(r)}$$

Thm Assumptions as above, we have

$$u(\tau) = \varepsilon_K^{-h_K}\left(1 - \sqrt{p}\, q + \dots\right)$$

$$\breve{u}(\tau) = q^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty} (1-q^n)^{\chi(n)}$$

where h_K is the class number and $\varepsilon_K > 1$ the fundamental unit of the real quadratic field $K = \mathbb{Q}(\sqrt{p})$.

(Here as before, $\breve{u} = u \circ W_p$, where W_p is the Fricke involution.)

Example If $p = 5$ then $h_K = 1$ and $\varepsilon_K = \frac{1+\sqrt{5}}{2}$, so

$$\eta(\tau) = \frac{-1+\sqrt{5}}{2}\left(1 - \sqrt{5}\, q + \cdots\right).$$

Remark Gauß defined h_K as the number equivalence classes of

$$Q(X,Y) = AX^2 + BXY + CY^2 \in \mathbb{Z}[X,Y]$$

of discriminant $B^2 - 4AC = p$, where $Q_1(X,Y) \sim Q_2(X,Y)$ if we may tranform Q_1 into Q_2 via a linear change of coordinates $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Thm We have the limit

$$\lim_{t\to 0^+} \frac{\sum_{n=0}^{\infty} p_+(n)\, e^{-2\pi n t}}{\sum_{n=0}^{\infty} p_-(n)\, e^{-2\pi n t}} = \varepsilon_K^{h_K}$$

Proof

$$\frac{1}{\check{u}} = q^{-\frac{1}{2}B_{2,\chi}} \frac{\prod_{m\in S_+} \frac{1}{1-q^m}}{\prod_{m\in S_-} \frac{1}{1-q^m}} = q^{-\frac{1}{2}B_{2,\chi}} \frac{\sum_{n=0}^{\infty} p_+(n)\, q^n}{\sum_{n=0}^{\infty} p_-(n)\, q^n} ;$$

$$\lim_{t\to 0^+} \check{u}(it) = \lim_{t\to 0^+} u(W_p(it)) = \lim_{t\to\infty} u(it) = \varepsilon_K^{-h_K} \qquad \square$$

The above is a generalization of a limit formula due to Schur for $p = 5$, which may be regarded as a consequence of the Rogers-Ramanujan continued fraction identity

$$\check{u}(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{\ddots}}}}$$

<u>Thm</u> (Petersson) If p is as above, then

$$\lim_{n \longrightarrow \infty} \frac{p_+(n)}{p_-(n)} = \varepsilon_K^{h_K},$$

Here as before, $p_\pm(n)$ is the # of partitions

$$n = \lambda_1 + \dots + \lambda_k, \quad \lambda_1 \geq \dots \geq \lambda_k,$$

s.t. $\chi(\lambda_i) = \pm 1$.