$\mathbb{Z}$ Dirichlet Series

Lemma 1.

Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and let $\{f_n\}$ be a sequence of holomorphic functions on $\mathcal{U}$ which converges uniformly on every compact set to a function f. Then f is holomorphic in $\mathcal{U}$ and the derivatives f_n' of the f_n converge uniformly on all compact subsets to the derivative f' of f.

Proof.

Recall the Cauchy Integral formula. Let D be a closed disc contained in $\mathcal{U}$ and let $\mathcal{C}$ be its boundary oriented as usual, then since every $f_n(z)$ is holomorphic, it is completely determined by it's values in the boundary of D.

$z_0 \in D$

$$f_n(z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f_n(z)}{z - z_0}\, dz$$

Taking the limit we get

$$f(z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz.$$

thus f is holomorphic

Use Cauchy's differentiation formula

$z_0 \in D$
$$f_n'(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f_n(z)}{(z-z_0)^2}\,dz$$

Proceed as in the later case

$$f'(z_0) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{(z-z_0)^2}\,dz$$

So the sequence of derivatives converge to the derivative of the limit.

Lemma 2

Let $\{a_n\}, \{b_n\}$ be two sequences. Define

$$A_{m,p} = \sum_{n=m}^{n=p} a_n \quad \text{and} \quad S_{m,m'} = \sum_{n=m}^{n=m'} a_n b_n$$

Then

$$S_{m,m'} = \sum_{n=m}^{n=m'-1} A_{m,n}(b_n - b_{n+1}) + A_{m,m'} b_{m'}$$

Proof:

Write $a_n = A_{m,n} - A_{m,n-1}$

$$\Rightarrow S_{m,m'} = a_m b_m + \sum_{n=m+1}^{m'} (A_{m,n} - A_{m,n-1}) b_n$$

$$= a_m b_m - (A_{m,m} - A_{m,m-1}) b_m + \sum_{n=m}^{m'} (A_{m,n} - A_{m,n-1}) b_n$$

$$= a_m b_m - a_m b_n + (A_{m,m'} - A_{m,m'-1}) b_{m'} + \sum_{n=m}^{m'-1} (A_{m,n} - A_{m,n-1}) b_n$$

$$= A_{m,m'} b_{m'} + \sum_{n=m}^{m'-1} A_{m,n} b_n - \sum_{n=m}^{m'-1} A_{m,n-1} b_n - A_{m,m'-1} b_{m'}$$

$$= A_{m,m'} b_{m'} + \sum_{n=m}^{m'-1} A_{m,n} b_n - \sum_{n=m}^{m'-1} A_{m,n} b_{n+1}$$

$$= A_{m,m'} b_{m'} + \sum_{n=m}^{m'-1} A_{m,n} (b_n - b_{n+1})$$

$\square$

Lemma 3

Let α, β be two real numbers with $0 < \alpha < \beta$. Let $z = x + iy$ with $x, y \in \mathbb{R}$ and $x > 0$. Then

$$|e^{-\alpha z} - e^{-\beta z}| \leq \left|\frac{z}{x}\right| (e^{-\alpha x} - e^{-\beta x})$$

Proof

$$|e^z| = e^x$$

Write
$$e^{-\alpha z} - e^{-\beta z} = z \int_\alpha^\beta e^{-tz}\, dt$$

then
$$|e^{-\alpha z} - e^{-\beta z}| \leq \left| z \int_\alpha^\beta e^{-tz}\, dt \right| \leq |z| \int_\alpha^\beta |e^{-tz}|\, dt = |z| \int_\alpha^\beta e^{-tx}\, dt$$

$$= \frac{|z|}{x} (e^{-\alpha x} - e^{-\beta x})$$

$\square$

(remember $x > 0$).

Let $\{\lambda_n\}$ be an increasing sequence of real numbers tending to $+\infty$. For simplicity, we suppose that λ_n are ≥ 0.

Definition

A Dirichlet series with exponents $\{\lambda_n\}$ is a series of the form

$$\sum a_n e^{-\lambda_n z} \qquad a_n \in \mathbb{C}, \ z \in \mathbb{C}.$$

Examples

- Take $\lambda_n = \log n$, then we have the series $\sum \frac{a_n}{n^z}$
- $\lambda_n = n$, and setting $e^z = t$, $\sum a_n t^n$, a power series.

Proposition 6

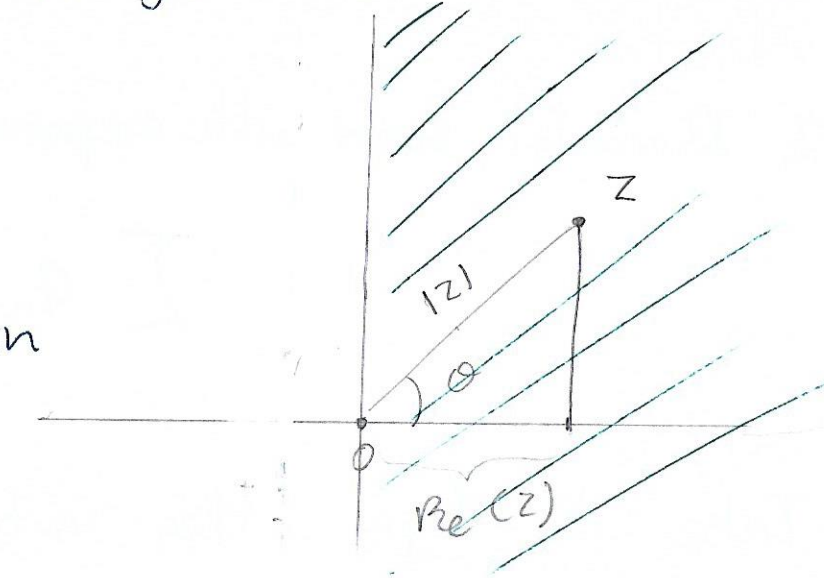
If the series $f(z) = \sum a_n e^{-\lambda_n z}$ converges for $z = z_0$, it converges uniformly in every domain of the form $\mathrm{Re}(z - z_0) \geq 0$, $\mathrm{Arg}(z - z_0) \leq \alpha$, $\alpha < \frac{\pi}{2}$

Proof

Translate z_0 to the origin, so $z_0 = 0$. Then the hypothesis means $\sum a_n$ converges.

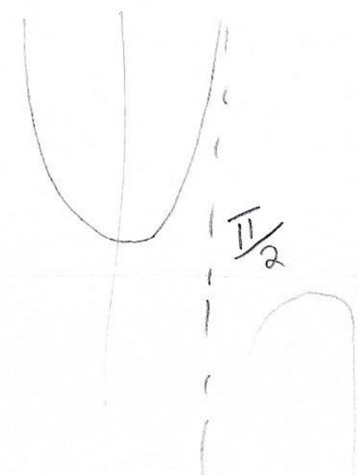
We must prove uniform convergence in the domain

$$\mathrm{Re}(z) \geq 0, \quad \frac{|z|}{\mathrm{Re}(z)} \leq K.$$



$$\frac{|z|}{\mathrm{Re}(z)} \leq K \Rightarrow \theta < \frac{\pi}{2}$$

↳ $\sec(\theta)$



Let $\varepsilon > 0$, since $\sum a_n$ convergence there is an N s.t if $m, m' \geq N$

$$|A_{m,m'}| \leq \varepsilon.$$

Apply Lemma 2 with $b_n = e^{-\lambda_n z}$

$$S_{m,m'} = \sum_{n=m}^{n=m'-1} A_{m,n}\left(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right) + A_{m,m'} e^{-\lambda_{m'} z}$$

Consider $z = x + iy$ and apply lemma 3:

$$|S_{m,m'}| \leq |A_{m,n}| \sum_{n=m}^{n=m'-1} |e^{-\lambda_n z} - e^{-\lambda_{n+1} z}| + |A_{m,m'}|$$

$$\leq \varepsilon \left(1 + \sum_{n=m}^{n=m'-1} \frac{|z|}{\operatorname{Re} z} \left(e^{-\lambda_n x} - e^{-\lambda_{n+1} x} \right) \right)$$

that is

$$|S_{m,m'}| \leq \varepsilon \left(1 + K \left(e^{-\lambda_m x} - e^{-\lambda_{m'} x} \right) \right)$$

So $S_{m,m'}$ is as small as we want, then the series converges uniformly.

Corollary 1

If f converges for $z = z_0$, it converges for the set $\operatorname{Re}(z) > \operatorname{Re}(z_0)$ and the function defined is holomorphic.

(Every function $e^{-\lambda_n z}$ is holomorphic)

Corollary 2

The set of convergence of the series f contains a maximal open half plane. (consider $\emptyset$ and $\mathbb{C}$ as half planes).

If the half plane of convergence is $Re(z) > \rho$, we say ρ is the abscissa of convergence of the series considered.

($\emptyset$ and $\mathbb{C}$ correspond to ∞ and $-\infty$).

Corollary 3

$f(z)$ converges to $f(z_0)$ when $z \longrightarrow z_0$ in the domain

$$Re(z - z_0) \geq 0, \quad |Arg(z - z_0)| \leq \alpha \qquad \alpha < \frac{\pi}{2}$$

This follows from the uniform convergence and the fact that $e^{-\lambda_n x}$ goes to $e^{-\lambda_n x_0}$ when $z \longrightarrow z_0$.

Corollary 4

The function $f(z)$ can be identically zero only if all its coefficients are zero.

- Show that $a_0 = 0$.

$$e^{\lambda_0 x} f(x) = a_0 + \sum_{n=1}^{\infty} a_n e^{(\lambda_0 - \lambda_n)} \qquad \lambda_0 - \lambda_n < 0$$

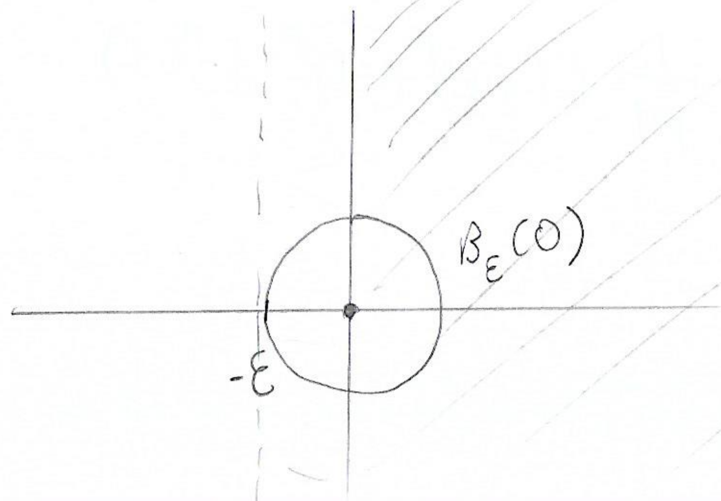
then $\lim_{x \to \infty} e^{\lambda_0 x} f(x) = a_0$, and if $f(x) = 0$, then

this implies $a_0 = 0$. We can do the same for $a_1, \dots, a_n$.

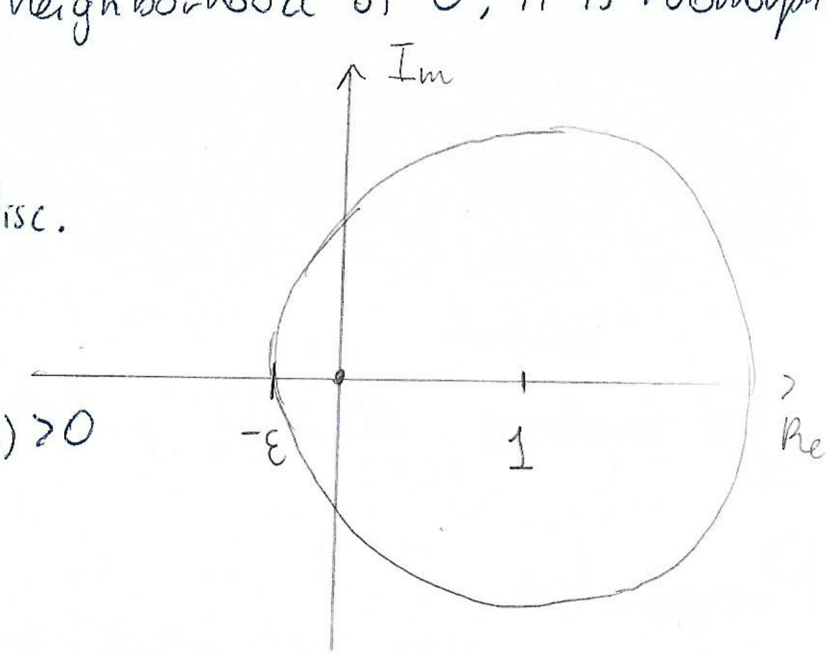
Proposition 7

Let $f(z) = \sum a_n e^{-\lambda_n z}$ be a Dirichlet Series whose coefficients $a_n \geq 0$ and $a_n \in \mathbb{R}$. Suppose it converges for $\mathrm{Re}(z) > \rho$, $\rho \in \mathbb{R}$, and that f can be extended analytically to a function holomorphic in a neighborhood of $z = \rho$. Then there exists a number $\varepsilon > 0$ s.t f converges for $\mathrm{Re}(z) > \rho - \varepsilon$.

Take $\rho = 0$



Since f is holomorphic for $\operatorname{Re}(z) > 0$ and in a neighborhood of 0, it is holomorphic in a disc $|z-1| \leq 1+\varepsilon$, $\varepsilon > 0$.



In particular, its Taylor Series converges in the disc.

By lemma I, the derivative of f is given by

$$f^{(p)}(z) = \sum a_n (-\lambda_n)^p e^{-\lambda_n z} \qquad \operatorname{Re}(z) > 0$$

hence $\quad f^{(p)}(1) = \sum a_n (-1)^p \cdot \lambda_n^p e^{-\lambda_n}$

So the Taylor Series for f can be written as

$$f(z) = \sum_{p=0}^{\infty} \frac{1}{p!} (z-1)^p f^p(1) \qquad |z-1| \leq 1+\varepsilon$$

In particular for $z = -\varepsilon$

$$f(-\varepsilon) = \sum_{p=0}^{\infty} \frac{1}{p!} (1+\varepsilon)^p (-1)^p f^p(1)$$

The series being convergent.

but $(-1)^p f^{(p)}(1) = \sum a_n \lambda_n^p e^{-\lambda_n}$ converges and has positive terms

Hence the double series with positive terms

$$f(-\varepsilon) = \sum_n \sum_p a_n \frac{1}{p!} (1+\varepsilon)^p \lambda_n^p e^{-\lambda_n}$$

converges. Rearranging the terms, we get

$$f(-\varepsilon) = \sum_n a_n e^{-\lambda_n} \sum_{p=0}^{\infty} \frac{1}{p!} (1+\varepsilon)^p \lambda_n^p \longrightarrow e^x \text{ series}$$

$$= \sum_n a_n e^{-\lambda_n} e^{\lambda_n (1+\varepsilon)} = \sum_n a_n e^{\lambda_n \varepsilon}$$

Then the Dirichlet Series converges for $z = -\varepsilon$, thus also for $\mathrm{Re}(z) > -\varepsilon$.

Ordinary Dirichlet Series

The case $\lambda_n = \log n$, corresponds to

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Prop 8

If the a_n are bounded, there is absolute convergence for $Re(s) > 1$

$$\sum |a_n| \left|\frac{1}{n^s}\right| \leq C \sum \left|\frac{1}{n^s}\right| = C \sum \frac{1}{n^x}$$

. this series converges for $x > 1$.

Prop 9.

If the partial sums $A_{m,p} = \sum_{m}^{p} a_n$ are bounded, there is convergence (not necessarily absolute) for $Re(s) > 0$

Assume $|A_{m,p}| \leq K$. and apply Abel's lemma

$$S_{m,m'} = \sum_{n=m}^{n=m'-1} A_{m,n}\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) + A_{m,m'}\frac{1}{(m')^s}$$

$$|S_{m,m'}| \leq K\left[\sum_{n=m}^{m'-1}\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| + \left|\frac{1}{(m')^s}\right|\right]$$

$$\leq K\left[\sum_{n=m}^{m'-1}\left(\left|\frac{1}{n^s}\right| + \left|\frac{1}{(n+1)^s}\right|\right) + \left|\frac{1}{(m')^s}\right|\right]$$

We can suppose s is real

$$|S_{m,m'}| \leq K\,2\sum_{n=m}^{m'}\frac{1}{(m)^s} \leq C\frac{1}{m^s}$$

then the convergence is clear.