$\mathbb{Z}$ Zeta function and L functions

Definition

A function $f : \mathbb{N} \longrightarrow \mathbb{C}$ is called multiplicative if $f(1) = 1$ and

$$f(mn) = f(m) f(n)$$

whenever m, n are relatively prime integers.

- Let f be a bounded multiplicative function

Lemma 4

The Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely for $Re(s) > 1$ and its sum in this domain is equal to the convergent infinite product

$$\prod_{p \in P} (1 + f(p) p^{-s} + \dots + f(p^m) p^{-ms} + \dots)$$

Proof

For the absolute convergence, use that $\forall n \in \mathbb{Z}$ $|f(n)| < K$

$$\sum \left| \frac{f(n)}{n^s} \right| \leq K \sum \left| \frac{1}{n^s} \right| = K \sum \frac{1}{n^x} < \infty$$

and the series converges for $x < 1$

- Let S be a finite set of prime numbers and $N(S) = \{ n \in \mathbb{Z}, n \geq 1 : n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} : p_1, \ldots, p_k \in S \}$

- $$\sum_{n \in N(S)} \frac{f(n)}{n^s} = \sum_{\alpha_1 \ldots \alpha_k} \frac{f(p_1^{\alpha_1}) \ldots f(p_k^{\alpha_k})}{p_1^{\alpha_1 s} \cdots p_k^{\alpha_k s}}$$ ← the product of all primes in S of power $m \in \mathbb{Z}$.

$$= \prod_{p \in S} \left(\sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}} \right)$$ ← producto de todos contra todos

- Make the finite set S bigger and bigger, then

$$\sum_{n \in N(S)} \frac{f(n)}{n^s} \longrightarrow \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

and thus $$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \in P} \left(\sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}} \right)$$

- Since we know the series converges for $Re(s) > 1$, then the infinite product also converges.

Lemma 5

If f is multiplicative in the strict sence (i.e. if $f(nm) = f(n)f(m)$ $\forall\, n, m \in \mathbb{N}$)

one has

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \in P} \frac{1}{1 - \frac{f(p)}{p^s}}$$

Proof : Now we have $f(p^m) = f(p)^m$ (it doesnt matter $p \mid p^m$)

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \in P} \left(\sum_{m=0}^{\infty} \frac{f(p^m)}{p^{ms}} \right) = \prod_{p \in P} \sum_{m=0}^{\infty} \left(\frac{f(p)}{p^s} \right)^m = \prod_{p \in P} \frac{1}{1 - \frac{f(p)}{p^s}} .$$

The Zeta function

Apply the preceding section with $f = 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - \frac{1}{p^s}}$$

for $\mathrm{Re}(s) > 1$

Prop 10

a) The zeta function is holomorphic and $\neq 0$ in the half plane $\operatorname{Re}(s) > 1$

b) $\zeta(s) = \frac{1}{s-1} + \phi(s)$ where $\phi(s)$ is holomorphic for $\operatorname{Re}(s) > 0$.

Proof

a) is clear by the previous lemmas.

b)
$$\int_1^\infty t^{-s}\,dt = \lim_{b\to\infty} \int_1^b t^{-s}\,dt = \lim \left.\frac{t^{-s+1}}{-s+1}\right|_1^b = 0 - \frac{1}{-s+1} = \frac{1}{s-1}$$

$$\sum_{n=1}^\infty \int_n^{n+1} t^{-s}\,dt = \frac{1}{s-1}$$

Hence we can write
$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^\infty \left(\frac{1}{n^s} - \int_n^{n+1} t^{-s}\,dt\right)$$
$$= \frac{1}{s-1} + \sum_{n=1}^\infty \int_n^{n+1} (n^{-s} - t^{-s})\,dt$$

Define $\quad \Phi_n(s) = \displaystyle\int_n^{n+1} (n^{-s} - t^{-s})\,dt \quad$ and $\quad \Phi(s) = \displaystyle\sum_{n=1}^{\infty} \Phi_n(s)$

We have to show Φ is defined and holomorphic for $\operatorname{Re}(s) > 0$.
Clearly each Φ_n is, so we just have to prove $\sum \Phi_n$ converges normally for $\operatorname{Re}(s) > 0$.

$$|\Phi_n(s)| \leq \sup_{n \leq t \leq n+1} |n^{-s} - t^{-s}| \quad (1), \quad \text{take } \Psi_n = n^{-s} - t^{-s}$$

Observe that in the unitary interval we have that if Ψ_n is continuous

$$\Psi_n(t') - \Psi_n(0) = \Psi_n'(c)\,(t' - 0) \qquad \text{for some } c \in (0, t')$$

$\Psi_n(t') = \sup\limits_{0 \leq t \leq 1} \Psi_n$

Since $\Psi_n(0) = 0$, $\quad |\Psi_n(t')| = |\Psi_n'(c)|\,|t'| \leq |\Psi_n'(c)| \leq \sup\limits_{0 \leq t \leq 1} |\Psi'(t)|$

Since $\Psi_n'(t) = \dfrac{s}{t^{s+1}} \implies |\Phi_n(s)| \leq \left|\dfrac{s}{n^{s+1}}\right| = \dfrac{|s|}{n^{x+1}}$

And thus the series converges normally for $\operatorname{Re}(s) \geq \varepsilon \quad \forall\, \varepsilon > 0$.

$$\sum \sup_{Re(s)>0} \phi_n(s) \leq \sum \frac{|s|}{n^{x+1}} < \infty \quad \text{for } x > \varepsilon.$$

- Normal convergence uniform convergence.

Corollary 1
The zeta function has a simple pole for $s=1$.

Corollary 2
When $s \to 1$, $\sum_p \frac{1}{p^{-s}} \sim \log \frac{1}{s-1}$ and $\sum_p \sum_{k\geq 2} \frac{1}{p^{ks}}$

remains bounded.

$$\zeta(s) = \prod_{p\in P} \frac{1}{1-\frac{1}{p^s}}$$

$$\log(\zeta(s)) = -\sum_p \log(1-p^{-s})$$

$$= -\sum_p \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-p^{-s})^n}{n}$$

$$= \sum_p \sum \frac{(-1)^{2n}(p)^{-sn}}{n} = \sum_{p\in P,\, n\geq 1} \frac{1}{n p^{ns}}$$

- $\log(\zeta(s)) = \sum_{p \in P} \frac{1}{p^s} + \sum_{p \in P,\, n \geq 2} \frac{1}{n\, p^{ns}}$

- Set $\Psi(s) = \sum_{p \in P,\, n \geq 2} \frac{1}{n\, p^{ns}}$, this series is majorized by the series

$$\sum \frac{1}{p^{ns}} \leq \sum \frac{1}{p^{2s} - p^s} = \sum \frac{1}{p^s(p^s - 1)} \leq \sum \frac{1}{p(p-1)}$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

$$= 1$$

Then $\Psi(s)$ is bounded, and as $s \longrightarrow 1$ we have that $\log(\zeta(s)) \sim \log\left(\frac{1}{s-1}\right)$, thus $\sum_{p \in P} \frac{1}{p^s} \sim \log\left(\frac{1}{s-1}\right)$.

Remark //

$\zeta(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with a single pole at $s=1$.

The function $\xi(s) = \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$ is menomorphic and satisfies the <u>functional equation</u> $\xi(s) = \xi(1-s)$. Moreover the zeta function satisfies

$$\zeta(-2n) = 0 \qquad \text{if } n > 0$$

$$\zeta(1-2n) = \frac{(-1)^n B_n}{2n} \qquad \text{if } n > 0.$$

wher B_n denotes the nth Bernoulli number.

- One conjectures (Riemann hypothesis) that the other zeros of ζ are on the line $\operatorname{Re}(s) = \frac{1}{2}$. This has been verified numerically for a large number of them ($>$ 3 million).

3.3 The L-functions

Let m be an integer ≥ 1. Remember that a character modulo m can be viewed as a function, defined on the set of integers prime to m. s.t. $\chi(ab) = \chi(a)\chi(b)$.

We can extend the function on $\mathbb{Z}$, by defining $\chi(a) = 0$ when $a | m$

The corresponding L function is defined by the Dirichlet Series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Prop 11

For $\chi = 1$ one has $L(s,1) = F(s)\zeta(s)$ with $F(s) = \prod_{p \mid m} (1-p^{-s})$

Proof

$$L(s,1) = \sum_{\substack{n=1 \\ n \nmid m}}^{\infty} \frac{1}{n^s} = \prod_{p \nmid m} \frac{1}{1-\frac{1}{p^s}} \quad (1)$$

$$= \prod_{p \nmid m} \left(\frac{1}{1-\frac{1}{p^s}}\right) \prod_{p} \frac{1-\frac{1}{p^s}}{1-\frac{1}{p^s}}$$

$$= \prod_{p \in \mathbb{P}} \frac{1}{1-\frac{1}{p^s}} \cdot \prod_{p \mid m} \left(1-\frac{1}{p^s}\right)$$

$$= \zeta(s) \prod_{p \mid m} \left(1-\frac{1}{p^s}\right)$$

$L(s,1)$ extends analytically for $\mathrm{Re}(s) > 0$ and has a simple pole at $s=1$.

Prop. 12

For $\chi \neq 1$ the series $L(s, \chi)$ converges in the half plane $\underbrace{Re(s) > 0}_{\text{actually } Re(s) > 1}$ one has

$$L(s, \chi) = \prod_{p \in P} \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

It is shown in the section corresponding for multiplicative functions. It remains to prove convergence of the series for $Re(s) > 0$. For that we will use Proposition 4 of Daniel's presentation. and Proposition 9 of yesterdays presentation.

Is enough to show

$$\sum_{n=u}^{v} \chi(n) < K \qquad u \leq v,$$

But these sums are zero or $\phi(m)$ (order of the group), so

$$\left| \sum_{n=u}^{v} \chi(n) \right| \leq \phi(m) .$$

Remark//

In particular $L(1,\chi)$ is finite when $\chi \neq 1$. The essential proof of Dirichlet's proof consists in showing that $L(1,\chi)$ is different from zero. (Section 4)

3.4) Product of L-functions relative to the same integer m.

$m \in \mathbb{Z}$, $m \geq 1$. If $p \nmid m$, we denote its image in $G(m) = (\mathbb{Z}/m\mathbb{Z})^*$ by $\tilde{p}$ and $f(p)$ the order of $\tilde{p}$ in $G(m)$. That is, $f(p)$ is the smallest integer $f \geq 1$ s.t $p^f \equiv 1 \pmod{m}$.

Put $g(p) = \dfrac{\phi(m)}{f(p)}$ (order of the quotient of $G(m)$ by the subgroup generated by $\tilde{p}$)

Lemma 6

If $p \nmid m$, one has the identity

$$\prod (1 - \chi(p)T) = (1 - T^{f(p)})^{g(p)}$$

where the product is taken over all characters of $G(m)$.

First, take W, the set of $f(p)$- roots of unity $\quad w \in W \Rightarrow w^{f(p)} = 1$

Take the identity

$$\prod_{w \in W} (1 - wT) = 1 - T^{f(p)}$$

then $\forall w \in W$, there are $g(p)$ characters of $G(m)$ s.t $\chi(\tilde{p}) = w$,

So in orther to consider all caracters I just have to mwltiply by $1 - T^{f(p)}$

$g(p)$ times

$$\prod_{\chi} (1 - \text{[illegible]} T) = \Big(\prod_{w \in W} (1 - wT) \Big)^{g(p)} = \left(1 - T^{f(p)}\right)^{g(p)}.$$

Define a new function

$$\zeta_m(s) = \prod_{\chi} L(s, \chi)$$

Proposition 13

One has $\zeta_m(s) = \prod_{p \nmid m} \dfrac{1}{\left(1 - \frac{1}{p^{f(p)s}}\right)^{g(p)}}$

This is a Dirichlet Series, with positive integral coefficients, converging in the half plane $Re(s) > 1$.

$$\zeta_m(s) = \prod_{\chi} L(s, \chi)$$

$$= \prod_{\chi} \left(\prod_{p \in P} \frac{1}{1 - \frac{\chi(p)}{p^s}} \right) \qquad (\text{if } p \mid m,\ \chi(p) = 0)$$

$$= \prod_{\chi} \left(\prod_{p \nmid m} \frac{1}{1 - \frac{\chi(p)}{p^s}} \right) = \prod_{p \nmid m} \prod_{\chi} \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

$$= \prod_{p \nmid m} \left(\frac{1}{1 - \frac{1}{p^{f(p)s}}} \right)^{g(p)}$$

The convergence is clear.

Thm 1

a) ζ_m has a simple pole at $s=1$

b) $L(1,\chi) \neq 0 \ \forall \ \chi \neq 1$.

b) $\Rightarrow$ a)

If $L(1,\chi) \neq 0 \ \forall \ \chi \neq 1$, the fact that $L(s,1)$ has a simple pole at $s=1$ shows that the same is true for ζ_m.

a) $\Rightarrow$ b)

By contradiction... Pending.