Lecture 17 Quadratic forms

Given a commutative ring R and an R-module V, a quadratic form on V is a function $Q : V \longrightarrow R$ s.t.

(i) $\forall a \in R, v \in V$

$$Q(ax) = a^2 Q(x)$$

(ii) the function

$$V \times V \longrightarrow R$$

$$(x,y) \longmapsto Q(x+y) - Q(x) - Q(y)$$

is a bilinear form. We call (V, Q) a quadratic module.

For this lecture we shall assume $R \in k$, where k is a field s.t. $\chi(k) \neq 2$ so that we may define the symmetric bilinear form

$$x \cdot y := \frac{1}{2}\Big(Q(x+y) - Q(x) - Q(y) \Big),$$

which is s.t. $\forall\, x \in V$

$$Q(x) = x \cdot x,$$

thus giving a bijective correspondence between quadratic forms on V and symmetric bilinear forms on V.

Given quadratic modules (V, Q) and (V', Q'), a morphism

$$(V, Q) \xrightarrow{f} (V', Q')$$

is a linear map $V \xrightarrow{f} V'$ s.t. $\forall\, x, y \in V$

$$f(x \cdot y) = f(x) \cdot f(y).$$

If $\mathcal{B} = \{e_1, \dots, e_n\}$ is a basis for V, then the matrix $[Q]_{\mathcal{B}}$ of Q with respect $\mathcal{B}$ is

$$[Q]_{\mathcal{B}} := \left(e_i \cdot e_j\right)_{i,j=1}^{n}$$

Note that $\forall\, x \in V$ we have

$$Q(x) = \sum_{i,j=1}^{n} (e_i \cdot e_j)\, x_i x_j ,$$

where $[x]_{\mathcal{B}} = (x_1, \dots, x_n)$. If we change the basis $\mathcal{B}$ via $M \in GL_n(\mathbb{R})$ then

$$[Q]_{\mathcal{B}'} = M\,[Q]_{\mathcal{B}}\,M^t$$

where $M\mathcal{B} = \mathcal{B}'$ is the new basis. Therefore

$$\det([Q]_{\mathcal{B}'}) = \det([Q]_{\mathcal{B}})\det(M)^2.$$

<u>Defn</u> The discriminant disc(Q) of a quadratic form Q is the image of $\det([Q]_{\mathcal{B}})$ in $R^\times / (R^\times)^2$ or 0.

<u>Defn</u> We say that $x, y \in V$ are orthogonal if $x \cdot y = 0$.

Given a subspace $W \leq V$ we write

$$W^0 := \{ x \in V \mid \forall y \in W : x \cdot y = 0 \} \leq V$$

an say that subspaces $V_1, V_2 \leq V$ are orthogonal if $V_1 \leq V_2^0$, and Q is nondegenerate if $V^0 = 0$, i.e. if $\mathrm{disc}(Q) \neq 0$.

__Proposition__ Suppose (V, Q) is nondegenerate. Then all morphisms

$$(V, Q) \xrightarrow{f} (V', Q')$$

are injective.

Proof

If f is as above and pick $x \in \mathrm{Ker}(f)$. Then $\forall\, y \in V$

$$x \cdot y = f(x) \cdot f(y) = 0.$$

The non degeneracy of Q implies that $x = 0$ $\square$

The binary quadratic form Q attached to $A, B, C \in \mathbb{R}$ is

$$Q := Ax^2 + Bxy + Cy^2 = [x, y] \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =: [A, B, C]$$

The set $\mathcal{Q}$ of such forms is a quadratic space. Indeed, just put

$$Q \longmapsto D(Q) := B^2 - 4AC = -4 \cdot \det \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix},$$

with symmetric bilinear form

$$Q \cdot Q' = BB' - 2(AC' + A'C),$$

where $Q = [A, B, C]$ and $Q' = [A', B' C'] \in \mathcal{Q}$.

We say $Q, Q' \in \mathcal{Q}$ are equivalent if $\exists\ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{R})$ s.t.

$$[A, B, C](\alpha x + \beta y, \gamma x + \delta y) = \begin{pmatrix} \alpha^2 & \alpha\gamma & \gamma^2 \\ 2\alpha\beta & \alpha\delta + \beta\gamma & 2\gamma\delta \\ \beta^2 & \beta\delta & \delta^2 \end{pmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Prop'n We have a group homomorphism

$$SL_2(\mathbb{R}) \xrightarrow{\rho} SL_3(\mathbb{R})$$

$$M \longmapsto \begin{pmatrix} \alpha^2 & \alpha\gamma & \gamma^2 \\ 2\alpha\beta & \alpha\delta + \beta\gamma & 2\gamma\delta \\ \beta^2 & \beta\delta & \delta^2 \end{pmatrix}^{-1} =: \rho_M$$

Moreover, its image is contained in $\mathrm{Aut}(\mathcal{Q}) \cong SO(1,2)$.

Here $SO(1,2)$ denotes the subgroup of orientation preserving automorphs of the Lorentz form

$$\mathcal{L} := x_1^2 + x_2^2 - x_3^2$$

from Special Relativity, with 2 space-like dimensions (and 1 time-like dimension). The light cone is the set of $P \in \mathbb{R}^3$ s.t. $\mathcal{L}(P) = 0$.