Lecture 18 Finiteness of the class number

From now on we shall assume that $R = \mathbb{Z}$ and put $\Gamma = PSL_2(\mathbb{Z})$.

The representation ρ gives an action of $SL_2(R)$ on $\mathcal{Q}$

$$\Gamma \times \mathcal{Q} \longrightarrow \mathcal{Q}$$

$$(M, Q) \longmapsto M \cdot Q := \rho_M(Q)$$

Clearly $\forall D_0 \in R$ this action is well-defined on

$$\mathcal{Q}_{D_0} := \{ Q \in \mathcal{Q} \mid D(Q) = D_0 \},$$

and also on

$$\mathcal{Q}_{D_0}^{+} := \{ [A, B, C] \in \mathcal{Q}_{D_0} \mid A > 0 \}$$

if $D_0 < 0$. We define the class number of D_0 as

$$h(D_0) := \begin{cases} \left| \mathcal{Q}_{D_0}^{+} / \Gamma \right|, & \text{if } D_0 < 0 \\ \left| \mathcal{Q}_{D_0} / \Gamma \right|, & \text{otherwise.} \end{cases}$$

Prop'n The class number $h(D_0)$ is finite.

Proof

Case 1 $D_0 < 0$: We shall see that any $Q = [A, B, C] \in \mathcal{Q}_{D_0}$ is equivalent to one s.t. either

$$\left.\begin{array}{l} -A < B \leq A < C \\ \text{or} \\ 0 \leq B \leq A = C \end{array}\right\} \star$$

For that purpose we define

$$\tau_Q := \frac{-B + \sqrt{B^2 - 4AC}}{2A} \in \mathcal{H} := \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}.$$

Recall that Γ acts on $\mathcal{H}$ via Möbius transformations

$$\mathcal{H} \xrightarrow{\mu_M} \mathcal{H}$$

$$\tau \longmapsto \frac{\alpha\tau+\beta}{\gamma\tau+\delta} = \mu_M(\tau) =: M\cdot\tau$$

where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, and we know that $\forall \tau \in \mathcal{H}$ $\exists!\, M \in SL_2(\mathbb{Z})$ s.t. $\mu_M(\tau) \in \mathcal{F}$ where

$$\mathcal{F} := \left\{ \tau \in \mathcal{H} \;\middle|\; \tfrac{1}{2} \leq \mathrm{Re}(\tau) < \tfrac{1}{2} \right\} \cap \left\{ \tau \in \mathcal{H} \;\middle|\; |\tau| > 1 \text{ or } \left(|\tau| = 1 \;\&\; \mathrm{Re}(\tau) \leq 0 \right) \right\}$$

We may see that $\forall$ $M \in SL_2(\mathbb{Z})$, $Q \in \mathcal{Q}_{D_0}$

$$M \cdot \tau_Q = \tau_{M \cdot Q}$$

where

$$\mathcal{Q}_{D_0}^{+} \longrightarrow \mathcal{H}$$

$$Q \longmapsto \tau_Q := \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

is an injective map of Γ-sets. So it suffices to show that $\tau_Q \in \mathcal{F} \iff (\cancel{A})$.

We shall first assume that $\tau_Q \in \mathcal{F}^{int}$, i.e.

(i) $|\tau_Q| > 1$

(ii) $|\mathrm{Re}(\tau_Q)| < \frac{1}{2}$

Condition (i) yields

$$\left(\frac{-B+\sqrt{B^2-4AC}}{2A}\right)\left(\frac{-B-\sqrt{B^2-4AC}}{2A}\right) > 1$$

$$\frac{B^2-(B^2-4AC)}{4A^2} > 1$$

$$C > A$$

Condition (ii) yields

$$\left| \frac{-B}{2A} \right| < \frac{1}{2}$$

$$|B| < A$$

The above steps are reversible. Hence

$$\tau_Q \in \mathcal{F}^{int} \iff |B| < A < C$$

The case τ_Q on the remaining of $\mathcal{F}$ is left as an easy exercise for the reader. We shall need the following.

Lemma If $D_0 < 0$ and $Q = [A, B, C] \in \mathcal{Q}^+_{D_0}$ is reduced, then

$$A < \left(\frac{|D_0|}{3}\right)^{\frac{1}{2}}.$$

Proof of lemma

Assume that $Q \in \mathcal{Q}^+_{D_0}$ is reduced. Then

$$|D_0| = 4AC - B^2 \geqslant 4A^2 - A^2 = 3A^2 \qquad \square$$

The $D_0 < 0$ case follows as $|B| \leqslant A \leqslant \left(\frac{|D_0|}{3}\right)^{\frac{1}{2}}$ and C is uniquely determined by A, B, D_0.

Case 2 $D > 0$: We shall see that any $Q = [A, B, C] \in \mathcal{Q}_D$ is equivalent to one s.t. either

$$|A + C| < \frac{B}{2}$$

or

$$A + C = -\operatorname{sign}(A) \frac{B}{2} .$$