Yesterday...

→ Theorem.

If $N > 0$, $\displaystyle\lim_{N\to\infty} \frac{H(N)}{N} = w \frac{\phi(|d|)}{|d|} \sum_{m=1}^{\infty} \left(\frac{d}{m}\right)\frac{1}{m}$.

where $\displaystyle H(N) = \sum_{\substack{1 \le n \le N \\ (n,d)=1}} R(n)$, $\displaystyle R(n) = w \sum_{m \mid n} \left(\frac{d}{m}\right)$.

$$\Leftrightarrow \quad \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n) = w \frac{\phi(|d|)}{|d|} \sum_{m=1}^{\infty} \left(\frac{d}{m}\right)\frac{1}{m}. \quad (13)$$

Since $\frac{\phi(|d|)}{|d|}$ measure the density of the integers n for which $(n,d)=1$, we can express the result in the form: The averge with respect to n of $R(n)$ is $w\,L(1,\chi)$, where $\chi(m) = (d|m)$.

→ The next step is to evaluate the averge of $R(n)$ from its original definition.

Let $R(n,f)$ denote the number of representations of n (primary if $d>0$) by a paticular form of discriminant d. Then

$$(14) \qquad R(n) = \sum_{f} R(n,f),$$

where the summation is over a representative set of forms (with $a>0$), so that the number of terms in the sum is $h(d)$.

We shall now evaluate

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n,f).$$

and it will turn out to be independent of f.

→ Comparison of the two limits will give the relation between $h(d)$ and $L(1,\chi)$.

Take first the case $d<0$. Then

$$\sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n,f)$$

is the number of pairs of integers x,y satisfying

$$0<ax^2+bxy+cy^2\le N, \qquad (ax^2+bxy+cy^2,d)=1.$$

The second condition limits x,y to certain pairs of residue classes to the modulus $|d|$. It turns out that the number of these pairs is $|d|\,\phi(|d|)$, and it is independent of the choice of the sets of residues.

Hence we consider the number of pairs of integers x,y satisfying

$$ax^2+bxy+cy^2 \leq N, \qquad x \equiv x_0 \pmod{|d|}, \; y \equiv y_0 \pmod{|d|}$$

The first inequality expresses that the point (x,y) is in an ellipse with center at the origin, and as $N \to \infty$ this ellipse expands uniformly.

The area of the ellipse is

$$\frac{2\pi}{\sqrt{4ac-b^2}} \cdot N = \frac{2\pi}{|d|^{\frac{1}{2}}} \cdot N$$

→ The number of lattice points in the ellipse is asymptotic to

$$\frac{1}{|d|^2} \frac{2\pi}{|d|^{1/2}} \cdot N \qquad \text{as} \quad N \to \infty$$

(we need a proof of this!).

We need to multiply this by $|d| \cdot \phi(|d|)$ to allow for the various possibilities for x_0, y_0.

Thus

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\(n,d)=1}}^{N} R(n,f) = \frac{1}{|d|^2} \cdot \frac{2\pi}{|d|^{1/2}} \cdot |d|\, \phi(|d|)$$

$$\Rightarrow \quad \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\(n,d)=1}}^{N} R(n,f) = \frac{\phi(|d|)}{|d|} \cdot \frac{2\pi}{|d|^{1/2}}$$

Therefore we have the two limits

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\(n,d)=1}}^{N} R(n) = w\frac{\phi(|d|)}{|d|} \sum_{m=1}^{\infty} \left(\frac{d}{m}\right)\frac{1}{m} \qquad (*)$$

$$\rightarrow \quad \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1\\(n,d)=1}}^{N} R(n,f) = \frac{\phi(|d|)}{|d|} \cdot \frac{2\pi}{|d|^{1/2}}.$$

The first limit (*) can be viewed using (14)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\(n,d)=1}}^{N} R(n) = \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\(n,d)=1}}^{N}\sum_{f} R(n,f)$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\(n,d)=1}}^{N} h(d)\,R(n,f) = h(d)\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\(n,d)=1}}^{N} R(n,f)$$

$$\Rightarrow h(d)\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\(n,d)=1}}^{N} R(n,f) = w\frac{\phi(|d|)}{|d|}L(1,\chi). \quad (**)$$

$$\rightarrow \lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n=1\\(n,d)=1}}^{N} R(n,f) = \frac{\phi(|d|)}{|d|}\,\frac{2\pi}{|d|^{1/2}}. \quad ($$

Comparing these two limits:

$$w\frac{\phi(|d|)}{|d|}L(1,\chi) = h(d)\frac{\phi(|d|)}{|d|}\,\frac{2\pi}{|d|^{1/2}}.$$

$$\Rightarrow h(d)\cdot\frac{2\pi}{|d|^{1/2}} = w\,L(1,\chi)$$

$$\Rightarrow \boxed{h(d) = \frac{w|d|^{1/2}}{2\pi}L(1,\chi) \quad d<0.} \quad (15)$$

* Now take the case $d>0$.

Arguing as before, we need the number of integer points (x,y) satisfying

$$ax^2+bxy+cy^2 \leq N, \quad x-\theta y > 0, \quad 1 \leq \frac{x-\theta' y}{x-\theta y} \leq \varepsilon^2$$

and

$$x \equiv x_0 \pmod d, \quad y \equiv y_0 \pmod d$$

The first of conditions represents a sector of a hyperbola bounded by two fixed (or rather, half-lines) through the origin.

We can calculate the area of this sector by changing the coordinates from x, y to ξ, η, where

$$\xi = x - \theta y, \quad \eta = x - \theta' y.$$

We have

$$\frac{\partial(\xi, \eta)}{\partial(x,y)} = \theta - \theta' = \frac{-b+\sqrt{d}}{2a} - \left(\frac{-b-\sqrt{d}}{2a}\right) = \frac{\sqrt{d}}{a}.$$

In the ξ, η plane, the sector is given by

$$\xi\eta \leq N/a \quad (\text{since we have } a(x-\theta y)(x-\theta' y))$$

$$\xi > 0, \quad 1 \leq \frac{\eta}{\xi} \leq \varepsilon^2 \Rightarrow 1 \leq \eta \leq \varepsilon^2 \xi.$$

These conditions are equivalent to

$$0 < \xi \leq (N/a)^{1/2}, \quad \xi \leq \eta < \min(\varepsilon^2 \xi, N/a\xi).$$

Hence the area is

$$\int_0^{\xi_1} (\varepsilon^2 \xi - \xi)\, d\xi + \int_{\xi_1}^{(N/a)^{1/2}} \left(\frac{N}{a\xi} - \xi\right) d\xi$$

where $\xi_1 = \varepsilon^{-1}(N/a)^{1/2}$.

$$= \varepsilon^2 \frac{\xi^2}{2} - \frac{\xi^2}{2}\Big|_0^{\xi_1} + \frac{N}{a}\log(\xi) - \frac{\xi^2}{2}\Big|_{\xi_1}^{(N/a)^{1/2}}$$

$$= \varepsilon^2\frac{\xi_1^2}{2} - \frac{\xi_1^2}{2} + \frac{N}{a}\log\left(\left(\frac{N}{a}\right)^{1/2}\right) - \frac{\left((N/a)^{1/2}\right)^2}{2} - \frac{N}{a}\log(\xi_1) + \frac{\xi_1^2}{2}$$

$$= \frac{\varepsilon^2\left(\varepsilon^{-1}(N/a)^{1/2}\right)^2}{2} + \frac{N}{2a}\log\left(\frac{N}{a}\right) - \frac{N}{2a} - \frac{N}{a}\log\left(\varepsilon^{-1}(N/a)^{1/2}\right)$$

$$= \frac{N}{2a} + \frac{N}{2a}\log\left(\frac{N}{a}\right) - \frac{N}{2a} - \frac{N}{a}\left[\log(\varepsilon^{-1}) + \frac{1}{2}\log\left(\frac{N}{a}\right)\right]$$

$$= \frac{N}{a}\log(\varepsilon).$$

This has to be divided by $\frac{\sqrt{d}}{a}$ to give the area in the x, y plane. We have then to divide this by d^2 to allow for the congruence to the modulus d, and to multiply by $d\phi(d)$ to allow for the choices of x_0, y_0.

This gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n,f) = \frac{\log(\varepsilon)}{a} \cdot \frac{a}{\sqrt{d}} \cdot \frac{1}{d^2} \cdot d\phi(d)$$

$$\Rightarrow \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n,f) = \frac{\phi(d)}{d} \frac{\log\varepsilon}{d^{1/2}}.$$

Comparison with (**)

$$h(d) \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ (n,d)=1}}^{N} R(n,f) = \frac{w\phi(d)}{d} L(1,\chi)$$

$$\Rightarrow \frac{h(d)\,\phi(d)}{d}\,\frac{\log \varepsilon}{d^{1/2}} = w\,\frac{\phi(d)}{d}\,L(1,\chi)$$

(since $w=1$ for $d>0$)

$$\Rightarrow \boxed{h(d) = \frac{d^{1/2}}{\log \varepsilon}\cdot L(1,\chi).} \quad (16)$$

This complete the first stage of the work.

$\rightarrow$ (15), (16) $\Rightarrow L(1,\chi) > 0$.

$\Rightarrow$ There remains the question of expressing $L(1,\chi)$ by means of a finite sum.

This involves an evaluation of a Gauss' sum.

1 Proposition: $$\sum_{m=1}^{|d|} \left(\frac{d}{m}\right) e^{\frac{2\pi i m n}{|d|}} = \left(\frac{d}{n}\right)\varepsilon\sqrt{d}.$$

where $\varepsilon = 1$ if $d>0$ and $\varepsilon = i$ if $d<0$.

(Proof: 5 pages in Landau's book!).

2 Proposition: For $0<\theta<2\pi$, we have

$$\sum_{n=1}^{\infty} \frac{\sin \theta n}{n} = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\sum_{n=1}^{\infty} \frac{\cos \theta n}{n} = -\log\left(2\sin\frac{\theta}{2}\right).$$

Theorem. If d is a fundamental discriminant, then

$$L(1,\chi) = -\frac{\pi}{|d|^{3/2}} \sum_{m=1}^{|d|} m \left(\frac{d}{m}\right) \text{ if } d<0,$$

$$L(1,\chi) = -\frac{1}{d^{1/2}} \sum_{m=1}^{d} \left(\frac{d}{m}\right) \log \sin \frac{\pi m}{d} \text{ if } d>0.$$

Proof: By the previous proposition, we have

$$\sqrt{d}\, L(1,\chi) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) \sqrt{d}\, \frac{1}{n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) e^{\frac{2\pi i m n}{|d|}}$$

(since $\sum_{n=1}^{\infty} < \infty$)

$$= \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \sum_{n=1}^{\infty} \frac{1}{n} e^{\frac{2\pi i m n}{|d|}}.$$

by $\cdot 2$, since $0 < \frac{2\pi m}{|d|} < 2\pi$.

1) Let $d>0$.

$$\sqrt{d}\; L(1,\chi) = \sum_{m=1}^{d} \left(\frac{d}{m}\right) \sum_{n=1}^{\infty} \frac{\cos\left(n\, \frac{2\pi m}{d}\right)}{n} = -\sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \log\left(2 \sin \frac{\pi m}{d}\right)$$

$$= -\sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \log \sin \frac{m\pi}{d}$$

since

$$\log 2 \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) = 0.$$

$$\Rightarrow \quad L(1,\chi) = -\frac{1}{d^{1/2}} \sum_{m=1} \left(\frac{d}{m}\right) \log \sin \frac{\pi m}{d} \quad \text{if } d>0$$

2) Let $d<0$

$$|d|^{1/2} L(1,\chi) = \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \sum_{n=1}^{\infty} \frac{\sin\left(n \frac{2\pi m}{|d|}\right)}{n}$$

$$= \sum_{m=1}^{|d|} \left(\frac{d}{m}\right)\left(\frac{\pi}{2} - \frac{\pi m}{|d|}\right)$$

$$= \sum_{m=1}^{|d|} \left(\frac{d}{m}\right)\frac{\pi}{2} - \frac{\pi}{|d|} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) m.$$

$$= \frac{-\pi}{|d|} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) m.$$

$$\Rightarrow \quad L(1,\chi) = \frac{-\pi}{|d|^{3/2}} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) m. \quad \blacksquare$$

Thus, the class number

$$\boxed{\begin{array}{ll} h(d) = -\dfrac{1}{\log \varepsilon} \displaystyle\sum_{m=1}^{d} \left(\frac{d}{m}\right) \log \sin \frac{\pi m}{d} & \text{if } d>0 \\[2ex] h(d) = -\dfrac{w}{2|d|} \displaystyle\sum_{m=1}^{|d|} m \left(\frac{d}{m}\right) & \text{if } d<0 \end{array}}$$

→ The case when $d=-q$, q prime $\equiv 3 \pmod 4$.

Assume $q>3$, then $w=2$.

$$\Rightarrow \boxed{h(d) = -\frac{1}{|d|} \sum_{m=1}^{q-1} m \left(\frac{-q}{m}\right) = -\frac{1}{q} \sum_{m=1}^{q-1} m \left(\frac{m}{q}\right).}$$