Lecture 21 Dedekind's ζ-function and reciprocity

Let K be a number field, L/K a finite field extn, and

$$\left.\begin{array}{llll} \mathcal{O}_L & \mathcal{O}_L\mathfrak{p} = \mathfrak{P}_1^{e_1}\cdot\ldots\cdot\mathfrak{P}_g^{e_g} & \text{uniquely} \\ | & | & \\ \mathcal{O}_K & \mathfrak{p} & \text{non-zero prime ideal of } \mathcal{O}_K \end{array}\right\} *$$

We have $[L:K] = \sum_{i=1}^{g} e_i f_i$, where

$$f_i = [\mathcal{O}_L/\mathfrak{P}_i : \mathcal{O}_K/\mathfrak{p}]$$

and $e_i = 1$ for almost all prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$.

From now on let's assume that L/K is Galois. Then

$\forall i \in \{1, \dots, g\} \quad \exists \sigma \in G(L/K)$ s.t.

$$\mathfrak{P}_i = \sigma(\mathfrak{P}_1).$$

So $(*)$ gives

$$\mathcal{O}_L \mathfrak{p} = (\mathfrak{P}_1 \cdot \ldots \cdot \mathfrak{P}_g)^e$$

and

$$[L:K] = efg,$$

where $e = e_1$ and $f = f_1$.

Given any number field K, the Dedekind ζ-function $\zeta_K(s)$ is the Dirichlet series

$$\zeta_K(s) := \sum_{0 \neq \mathfrak{a} \text{ ideal of } \mathcal{O}_K} N(\mathfrak{a})^{-s} =: \sum_{n=1}^{\infty} \frac{r_K(n)}{n^s}$$

The fact that $\mathcal{O}_K$ is a Dedekind domain is equivalent to

$$\zeta_K(s) = \prod_{0 \neq \mathfrak{p} \text{ prime ideal of } \mathcal{O}_K} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1}.$$

Th'm Suppose $K = \mathbb{Q}(\sqrt{q^*})$, where $q^* := (-1)^{\frac{q-1}{2}} q$, with q an odd prime. Then

$$\zeta_K(s) = \zeta(s) \, L(\chi, s) \qquad \star$$

where $L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ and $\chi(n) := \left(\frac{n}{q}\right)$. So

$$r_K(n) = \sum_{0 < d \mid n} \chi(d) \qquad \star'$$

for each $n \in \{1, 2, \dots\}$.

<u>Remark</u> It is possible define $r_K(0) \in \mathbb{Q}^\times$ so that

$$f_K(\tau) = \sum_{n=0}^{\infty} r_K(n) q^n$$

is the Fourier expansion of a modular form of weight 1.

((Cf. Deligne - Serre thm.)

Proof of this

We have $\chi(n n') \equiv (n n')^{\frac{q-1}{2}} \pmod{q}$,

$$\chi(n n') \equiv \chi(n)\chi(n') \pmod{q},$$

$$\chi(n n') = \chi(n)\chi(n'),$$

$\forall\ n, n' \in \mathbb{Z}$. Therefore

$$L(\chi, s) = \prod_{p \text{ prime}} (1 - \chi(p) p^{-s})^{-1}.$$

But $\forall$ prime ideal $\mathfrak{p} \overset{\neq 0}{\subseteq} \mathcal{O}_K$, we have $\mathcal{N}(\mathfrak{p}) = p^f$,

and $efg = 2$, with $p\mathcal{O}_K = \mathfrak{p}_1 \cdots \mathfrak{p}_g$. Therefore

$$\zeta_K(s) = \prod_{\substack{p \text{ prime} \\ e_p = 2}} (1 - p^{-s})^{-1} \prod_{\substack{p \text{ prime} \\ g_p = 2}} (1 - p^{-s})^{-2} \prod_{\substack{p \text{ prime} \\ f_p = 2}} (1 - p^{-2s})^{-1}$$

$$= \prod_{\left(\frac{q^*}{p}\right) = 0} (1 - p^{-s})^{-1} \prod_{\left(\frac{q^*}{p}\right) = 1} (1 - p^{-s})^{-2} \prod_{\left(\frac{q^*}{p}\right) = -1} (1 - p^{-s})^{-1} (1 + p^{-s})^{-1}$$

$$= \zeta(s) \prod_{p \text{ prime}} \left(1 - \left(\frac{q^*}{p}\right) p^{-s}\right)^{-1} = \zeta(s) \, L(\chi, s),$$ as

the QRL says that

$$\left(\frac{p}{q}\right) = \left(\frac{q^*}{p}\right).$$

Indeed,

$$\left(\frac{p}{q}\right) = \left(\frac{q^*}{p}\right) = \left(\frac{(-1)^{\frac{q-1}{2}} q}{p}\right) = \left(\frac{(-1)^{\frac{q-1}{2}}}{p}\right)\left(\frac{q}{p}\right) =$$

$$(-1)^{\frac{q-1}{2} \cdot \frac{p-1}{2}} \left(\frac{q}{p}\right) \quad \square$$