Lecture 22 Further remarks

Recall that, in particular, for each prime $p \equiv 1 \pmod 4$

$$L(\chi, 1) = h_K \cdot \frac{2 \log \varepsilon}{\sqrt{p}}$$

and $h_K \cdot 2 \log \varepsilon = - \sum_{a \in \mathbb{F}_p} \chi(a) \log |1 - \zeta^a|$, so

$$\varepsilon^{-2h_K} = \prod_{a \in \mathbb{F}_p} (1 - \zeta^a)^{\chi(a)}. \qquad \star$$

The latter may be found in Darmon, H., Stark–Heegner points over real quadratic fields, Contemporary Mathematics 210, AMS, 1998.

Here Darmon motivates his approach to the Stark Conjectures on the explicit generation of all abelian extensions E/K [1], where $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, e.g. the Hilbert class field, i.e.

$$\begin{array}{c} E \\ h_K \Big| \quad \text{maximal unramified abelian ext}^{n} \\ K \end{array}$$

Here as usual h_K denotes the class number of K.

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1 Hilbert's 12^{th} problem for K

Formula ($\star$) may also be applied to get information of the modular unit of Ogg and Ligozat. Let

$$u(\tau) := \prod_{r=1}^{\frac{p-1}{2}} \mathfrak{k}_{(0,r/p)}(\tau)^{\chi(r)},$$

where $\mathfrak{k}_a(\tau) = \mathfrak{k}(z, L_\tau)$, $a = (a_1, a_2) \in \mathbb{R}^2$ is determined by $z = a_1\tau + a_2$, $L_\tau := \mathbb{Z}\tau \oplus \mathbb{Z}$, $\tau \in \mathcal{H}$. Here

$$\mathfrak{k}(z, L) := e^{-\frac{1}{2}\eta(z,L)z}\sigma(z, L)$$

is the *Klein form*, σ is the Weierstraß sigma, and $z \mapsto \eta(z,L)$ gives the quasi-periods of the Weierstraß-zeta function attached to the lattice $L \subseteq \mathbb{C}$.

Let $\breve{u} := u \circ w_p$, where w_p is the Fricke involution

$$\begin{aligned} \mathcal{H} &\longrightarrow \mathcal{H} \\ \tau &\longmapsto -\frac{1}{p\tau} \end{aligned}$$

<u>Prop'n</u> (c) We have the Fourier expansions

$$\breve{u}(\tau) = q^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty} (1-q^n)^{\chi(n)} \qquad (1)$$

$$u(\tau) = \varepsilon_K^{-k_K} (1 - \sqrt{p}\, q + \dots) \qquad (2)$$

Proof

Consider Dedekinds η - function $\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau})$ and

Siegel's function $g_a(\tau) = \mathfrak{k}_a(\tau)\, \eta(\tau)^2$, which is s.t.

$$g_a(\tau) = -q^{\frac{1}{2} B_2(a_1)} e^{2\pi i a_2 (a_1 - 1)/2} (1 - q_z) \prod_{n=1}^{\infty} (1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}),$$

where $B_2(X) := X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial,

$q_z := e^{2\pi i z}$, $z \in \mathbb{C}$. Put $\zeta_p := e^{2\pi i / p}$ and note that

$$u(\tau) = \prod_{r=1}^{\frac{p-1}{2}} g_{(0,r/p)}(\tau)^{\chi(r)}$$

$$= \prod_{r=1}^{\frac{p-1}{2}} \left(\zeta_p^{-\frac{r}{2}} (1-\zeta_p^{r}) \prod_{n=1}^{\infty} (1 - q_\tau \zeta_p^{r})(1 - q_\tau \zeta_p^{-r}) \right)^{\chi(r)}$$

$$= \left(\prod_{r=1}^{\frac{p-1}{2}} \zeta_p^{-\chi(r)\frac{r}{2}} (1-\zeta_p^{r})^{\chi(r)} \right) \underbrace{\left(\prod_{n=1}^{\infty} \prod_{r=1}^{\frac{p-1}{2}} (1 - q_\tau \zeta_p^{r})^{\chi(r)} (1 - q_\tau \zeta_p^{-r})^{\chi(r)} \right)}_{f(\tau)}$$

$$= K \, f(\tau),$$

Ogg, Ligozat

So $(\star)$ implies that

$$K = \prod_{r=1}^{\frac{p-1}{2}} \left(\zeta_p^{-\frac{r}{2}} - \zeta_p^{\frac{r}{2}} \right)^{\chi(r)} = \varepsilon_K^{-h_K}.$$

But $f(\tau) = \prod_{n=1}^{\infty} \psi(q^n)$, where

$$\psi(X) = \prod_{r=1}^{\frac{p-1}{2}} (1 - X\zeta_p^r)^{\chi(r)} (1 - X\zeta_p^{-r})^{\chi(r)} = \prod_{r=1}^{\frac{p-1}{2}} (1 - X\zeta_p^r)^{\chi(r)}$$

$$\equiv 1 - S_p X \pmod{X^2},$$

Gauß sum $S_p = \sqrt{p}$.

Therefore $x(\tau) = \varepsilon_K^{-h_K} f(\tau) = \varepsilon_K^{-h_K}\left(1 - \sqrt{p}\, q + \cdots\right)$,

i.e. (2). To see (1) we use Kubert & Lang p. 27

$$t_{a\alpha}(\tau) = (c\tau + d)\, t_a(\alpha\tau), \quad \forall\ \alpha \in SL_2(\mathbb{Z}),$$

and we thus have for $\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ that

$$\check{x}(\tau) = \prod_{r=1}^{\frac{p-1}{2}} t_{(-r/p,\,0)}(p\tau)^{\chi(r)}$$

$$= \prod_{r=1}^{\frac{p-1}{2}} \left(q_\tau^{\frac{1}{2} B_2\left(\frac{r}{p}\right) p} \left(1 - q_\tau^r\right) \prod_{n=1}^{\infty} \left(1 - q_\tau^{pn+r}\right)\left(1 - q^{pn-r}\right) \right)^{\chi(r)}$$

$$= q_\tau^{\frac{1}{2}B_{2,\chi}} \prod_{r=1}^{\frac{p-1}{2}} \left(1 - q_\tau^{r}\right)^{\chi(r)} \prod_{r=1}^{\frac{p-1}{2}} \prod_{n=1}^{\infty} \left(1 - q_\tau^{pn+r}\right)^{\chi(r)} \left(1 - q_\tau^{pn-r}\right)^{\chi(r)}$$

$$= q_\tau^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty} \left(1 - q_\tau\right)^{\chi(r)}.$$

We have just shown (1) $\square$

<u>Remark</u> The function u is well-defined on

$$X_\chi(p) := \mathcal{H}^* / \Gamma_\chi(p),$$

where $\Gamma_\chi(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p) \;\middle|\; \text{s.t. } \chi(a) = 1 \right\}$.

This was used by Shimura in his work towards RM, which was another approach to Hilbert's 12^{th} problem for K.