Lecture 24 On the Birch and Swinnerton-Dyer conjecture

We may attempt to obtain information about $C(\mathbb{Q})$ by

(i) assuming that we have found a point $P \in C(\mathbb{Q})$, so that it has a Weierstraß eqn

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \quad (\star)$$

where WLOG we may assume that $a_1, \dots, a_6 \in \mathbb{Z}$.

It is a classical fact that $(\star)$ defines an elliptic curve E, which has a natural commutative group structure with identity element $O_E = (0:1:0)$.

(ii) studying the Hasse-Weil L-function of E

$$L(E,s) := \prod_{p \text{ prime}} \left(1 - a_p(E)\, p^{-s} + p^{1-2s}\right)^{-1}$$

$$=: \sum_{n=1}^{\infty} a_n(E)\, p^{-s},$$

where

$$a_p(E) := \begin{cases} p+1 - |E(\mathbb{F}_p)|, & \text{good red'n}, \\ -1, & \text{non-split mult red'n}, \\ 1, & \text{split mult red'n}, \end{cases}$$

with $|\Delta|$ minimal, assuming for simplicity E is semistable.

Here $\Delta(a_1, \dots, a_6)$ is the discriminant of the Weierstraß eqn (A). So in particular, $\forall$ p prime

$$p \mid \Delta \iff E \text{ has bad red'n at } p.$$

In fact, the minimality condition on $|\Delta|$ says that such Weierstraß eqn is a by-product of Tate's algorithm, which gives the best possible model for E.

<u>Th'm</u> (Hasse 1936) Notation as above, for each prime p,

$$|a_p(E)| \leqslant 2p^{\frac{1}{2}},$$

so the L-series $L(E,s)$ converges absolutely and uniformly on each compact subset of

$$\{ z \in \mathbb{C} \mid \mathrm{Re}(z) > \tfrac{3}{2} \}.$$

<u>Remark</u> As we shall see, Ramanujan's conjecture says that $\forall p$ prime

$$|\tau(p)| \leq 2p^{\frac{11}{2}}, = 2p^{\frac{k-1}{2}}$$

and, more generally the Ramanujan - Petersson conjecture says that

$$|a_p| \leqslant 2\, p^{\frac{k-1}{2}}, \quad \text{(Deligne's thm)}$$

where $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ is a Hecke eigenform WRT

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod N \right\}.$$

$(*)$

This lead to suspect that $\sum a_p(E)\, q^n$ is s.t. $(*)$, i.e. that elliptic curve is modular. Wiles and Taylor-Wiles proved this for all semistable elliptic curves $E/\mathbb{Q}$ 1994, and the general case was accomplished by Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor in 2001. A consequence of this fact is the functional eq'n for the Hasse-Weil L-function

$$\Lambda(E, 2-s) = -\varepsilon_N \Lambda(E, s), \qquad *$$

where $\varepsilon_N = \pm 1$ and

$$\Lambda(E, s) := N^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(E, s).$$

The fact that $L(E, s)$ is attached to a modular form $f(\tau) = \sum_{n=1}^{\infty} a_n(E) q^n$ of weight $k = 2$ s.t.

$$f|_2 w_N = \varepsilon_N f,$$

where w_N is the Fricke involution — the above notation will be explained later.

Another consequence is that $L(E,s)$ may be extended analitically to all the complex plane, so the following makes sense.

<u>Conjecture</u> (Birch & Swinnerton-Dyer) Put $r := \mathrm{rnk}(E(\mathbb{Q}))$. Then we have the Taylor expn around $s=1$,

$$L(E,s) = \kappa(s-1)^r + \cdots$$

where $\kappa \neq 0$. Moreover,

$$K = |\text{Ш}(E)|\, \Omega_E R_E \left(\prod_{p \mid N_E} c_p\right) \cdot |E(\mathbb{Q})_{\mathrm{tors}}|^{-2},$$

where $\text{Ш}(E)$ is the Shafarevich-Tate group, Ω_E is the real period attached to $(\ast)$, c_p is the Tamagawa number attached to p, and R_E is the regulator of E.

The first part of the B & S-D conjecture has been proven for $E/\mathbb{Q}$ s.t.

$$\operatorname{ord}_{s=1} L(E,s) = 0, 1,$$

by Gross - Zagier — they used the theory of Heegner divisors developed by Birch — and Kolyvagin.