Lecture 25 Basic definitions

As before, consider the Poincaré upper half plane

$$\mathcal{H} := \{ \tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0 \}$$

The 3-dimensional (non-compact), simple real Lie group

$$PSL_2(\mathbb{R}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; \det(g) = 1 \right\}$$

is isomorphic to the group $\mathrm{Aut}_{an}(\mathcal{H})$ of analytic automorphisms of $\mathcal{H}$, i.e.

$$PSL_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{Aut}_{an}(\mathcal{H})$$

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(\begin{array}{l} \mathcal{H} \longrightarrow \mathcal{H} \\ \tau \longmapsto \dfrac{a\tau + b}{c\tau + d} =: g \cdot \tau \end{array} \right)$$

Indeed, we may see that $\forall\, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$, $\forall\, \tau \in \mathbb{C}$, we have

$$\operatorname{Im}(g \cdot \tau) = \frac{\operatorname{Im}(\tau)}{|c\tau + d|^2},$$

so $PSL_2(\mathbb{R}) \cdot \mathcal{H} \subseteq \mathcal{H}$ and, moreover, the kernel (of the above group homomorphism) is clearly trivial and its image is $\mathrm{Aut}_{an}(\mathcal{H})$ by the Schwarz lemma, as

$$\mathcal{H} \overset{an}{\cong} \mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \}.$$

We have an embedding of topological groups

$$\begin{aligned} \mathbb{R} &\longrightarrow PSL_2(\mathbb{R}) \\ t &\longmapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = T^t \end{aligned}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus $PSL_2(\mathbb{R})$ is not compact. However, its finite-dimensional rep'n theory is equivalent to that of the compact Lie group $Spin(3) := \{ h \in \mathbb{H} \mid |h| = 1 \}$. The latter is the (universal) double covering group of group of rotations of $\mathbb{R}^3$,

$$Spin(3) \longrightarrow SO(3)$$

The modular group is the discrete subgroup

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R}) \;\middle|\; a, b, c, d \in \mathbb{Z} \right\}$$

If $\tau_1, \tau_2 \in \mathcal{H}$ are in the same Γ-orbit we write $\tau_1 \sim \tau_2$.

The Γ-orbit structure may be obtained with the help of the set

$$\mathcal{F} := \left\{ \tau \in \mathcal{H} \;\middle|\; |\tau| \geq 1 \;\&\; |\mathrm{Re}(\tau)| \leq \tfrac{1}{2} \right\},$$

which is the Dirichlet set with sides

$$\{ i\infty \to \rho \} + \{ \rho \to i \} + \{ i \to -\bar{\rho} \} + \{ -\bar{\rho} \to i\infty \}$$

Thm We have

(1) $\forall\, \tau \in \mathbb{H}\ \exists\, g \in \Gamma$ s.t. $g \cdot \tau \in \tilde{\mathcal{F}}$

(2) $\forall\, \tau_1, \tau_2 \in \tilde{\mathcal{F}}$: $\tau_1 \sim \tau_2$ then either

$$\mathrm{Re}(\tau_1) = \pm \frac{1}{2} \quad \& \quad \tau_1 = \tau_2 \pm 1$$

or

$$|\tau_1| = 1 \quad \& \quad \tau_1 = -\frac{1}{\tau_2}$$

(3) $\forall\, \tau \in \tilde{\mathcal{F}}$ the stabiliser $\Gamma_\tau = \{1\}$, except for

$\tau = i$: $\Gamma_\tau = \langle S \rangle$, so $|\Gamma_\tau| = 2$

$\tau = \rho$: $\Gamma_\tau = \langle ST \rangle$ " " $= 3$

$\tau = -\bar{\rho}$: $\Gamma_\tau = \langle TS \rangle$ " " $= 3$

Thm If $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then

$$\Gamma = \langle S, T \rangle,$$

i.e. $\forall\, g \in \Gamma$ we have

$$g = ST^{n_1} ST^{n_2} \cdot \ldots \cdot ST^{n_k},$$

for some $n_1, \ldots, n_k \in \mathbb{Z} - \{0\}$.

NB. We have already seen proofs of these thms in a previous course.

<u>Remark</u> We also have $\Gamma = \langle S, TS \rangle$, so we may see that Γ is the coproduct of C_2 and C_3, i.e. for any group G and group homomorphims $f_r : C_r \longrightarrow G$ with $r \in \{2,3\}$, $\exists!$ g'homomorphism $h : \Gamma \longrightarrow G$ s.t. the diagram

$$\begin{tikzcd}
& G & \\
& \Gamma \arrow[u, "h"] & \\
C_2 \arrow[ur] \arrow[uur, bend left, "f_2"] & & C_3 \arrow[ul] \arrow[uul, bend right, "f_2"']
\end{tikzcd}$$

commutes.

Fix $k \in \mathbb{Z}$. For each function $f : \mathcal{H} \longrightarrow \mathbb{C}$ and each $g \in GL_2(\mathbb{R})$ we define the slash operator at f by

$$(f|_k g)(\tau) := \det(g)^{\frac{k}{2}} (c\tau + d)^{-k} f(g \cdot \tau),$$

for each $\tau \in \mathcal{H}$. If f is meromorphic and s.t. $\forall\, g \in \Gamma$

$$f|_k g = g, \qquad \star$$

then we say that f is weakly modular of weight k.

Remark Note that $\forall\ g \in PSL_2(\mathbb{R})$

$$d(g\cdot\tau) = \frac{d\tau}{(c\tau+d)^2}.$$

Hence $(\bigstar)$ says that

$$f(g\cdot\tau)\, d(g\cdot\tau)^{\frac{k}{2}} = f(\tau)\, d(g\cdot\tau)^{\frac{k}{2}},$$

i.e. $f(\tau)\, d\tau^{\frac{k}{2}}$ is Γ-invariant.

From the second theorem we see that $f : \mathcal{H} \longrightarrow \mathbb{C}$ is weakly modular iff $\forall\ \tau \in \mathcal{H}$,

(i) $$f(\tau + 1) = f(\tau)$$

(ii) $$f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$$

From (i) we have a Fourier exp$^{\acute{}}$n

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where $q = e^{2\pi i \tau}$. We say that f is meromorphic at $i\infty$ if we have a Laurent expansion $f(\tau) = \dfrac{a_{-N}}{q^N} + \cdots$

and we say that f is holomorphic at $i\infty$ if the Fourier expansion of f is of the form

$$f(\tau) = a_0 + a_1 q + \cdots$$

We say that a weakly modular function is a modular function if it is meromorphic at $i\infty$. If f is holomorphic on $\mathcal{H}$ and also at $i\infty$, then we say that f is a modular form, and denote the $\mathbb{C}$-vector space of these functions by $M_k(\Gamma)$. We also define

$$S_k(\Gamma) := \{ f \in M_k(\Gamma) \mid a_0(f) = 0 \}$$

and call its elements cusp forms[1] of weight k, with respect to Γ.
The first non trivial cusp form is

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n,$$

as we shall see. But first we'll describe Eisenstein series.

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1 The $f \in S_k(\Gamma)$ is called a Spitzenform.