Lecture 26 Eisenstein series and Klein's j-function

A subgroup $\Lambda \subseteq \mathbb{C}$ is a *lattice* if

(i) $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ $\quad PSL_2(\mathbb{Z})$

(ii) $\mathbb{C} = \mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2$

for some $(\omega_1, \omega_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$, ordered WLOG s.t. $\frac{\omega_1}{\omega_2} =: \tau \in \mathcal{H}$.

In other words, $\Lambda \subseteq \mathbb{C}$ is s.t. $\mathbb{C}/\Lambda$ is a compact Riemann surface with universal covering $\mathbb{C} \longrightarrow \mathbb{C}/\Lambda$. Therefore the analytic isomorphism class $[\mathbb{C}/\Lambda]_{an}$ of $\mathbb{C}/\Lambda$ is determined by the homothety class $[\Lambda]_{\mathbb{C}^\times}$ of Λ and, conversely, $[\Lambda]_{\mathbb{C}^\times}$ determines $[\mathbb{C}/\Lambda]_{an}$.

Let $\mathcal{R} := \{ \Lambda \subseteq \mathbb{C} \mid \Lambda \text{ is a lattice} \}$ and note that

$$\begin{aligned} \mathcal{R}/\mathbb{C}^\times &\xrightarrow{\sim} \mathcal{H}/\Gamma \\ [\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2]_{\mathbb{C}^\times} &\longmapsto [\tau]_\Gamma \\ & \qquad\ \ \shortparallel \\ & \qquad\ \ \tfrac{\omega_1}{\omega_2} \end{aligned}$$

Now for each $k \in \mathbb{Z}$ we say that $F : \mathcal{R} \longrightarrow \mathbb{C}$ is of weight k if $\forall\ \lambda \in \mathbb{C}^\times, \Lambda \in \mathcal{R} : F(\lambda\Lambda) = \lambda^{-k} F(\Lambda)$. So $\exists!$ mod'r function f of w't k s.t.

$$F(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right).$$

Moreover,

$$\{ \mathcal{R} \xrightarrow{F} \mathbb{C} \text{ of weight } k \} \cong \{ \mathcal{H} \xrightarrow{f} \mathbb{C} \mid \text{modular function of weight } k \}$$

$$F \longmapsto f$$

We'll construct a $G_k : \mathcal{R} \to \mathbb{C}$ of weight k by letting

$$G_k(\Lambda) := \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^k} = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\omega_1 + n\omega_2)^k}$$

provided $k \in \{4, 6, 8, \ldots\}$ to have absolute convergence.

__Thm:__ We have a modular form of weight k

$$G_k(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^k}$$

s.t. $G_k(i\infty) = 2\zeta(k) \neq 0$, for each $k \in \{4, 6, 8, \ldots\}$.

The above non-cusp modular form is known as an Eisenstein series. They appear in the Laurent expn

$$\wp_{\Lambda_\tau}(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, G_{2k+2}(\tau)\, z^{2k},$$

where $\Lambda_\tau := \mathbb{Z}\tau \oplus \mathbb{Z}$, and

$$\wp_{\Lambda}(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

is the Weierstraß $\wp$-function. With the help of Liouville's thm we may see that

$$\mathbb{C}/\Lambda \xrightarrow{\sim} E(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C})$$

$$z+\Lambda \longmapsto \begin{cases} (\wp_\Lambda(z) : \wp'_\Lambda(z) : 1), & \text{if } z \notin \Lambda \\ (0:1:0), & \text{if } z \in \Lambda \end{cases}$$

where E is the non-singular cubic defined by homogenising

$$y^2 = 4x^3 - g_2 x - g_3,$$

when

$$g_2 := 60\, h_4(\tau),$$

$$g_3 = 140\, h_6(\tau).$$

Thus $\Delta(\tau) := g_2^3 - 27 g_3^2 \neq 0.$

We'll obtain the Fourier expansion of $G_k(\tau)$.

<u>Lemma</u> For each $k \in \{2,4,6,\dots\}$ we have the Fourier exp^n

$$\sum_{m \in \mathbb{Z}} \frac{1}{(m+\tau)^k} = \frac{1}{(k-1)!} (2\pi i)^k \sum_{r=1}^{\infty} r^{k-1} q^r, \quad \star$$

where $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$.

Proof

Recall that

$$\pi \cot \pi\tau = \frac{1}{\tau} + \sum_{m \in \mathbb{Z} - \{0\}} \left(\frac{1}{\tau + m} - \frac{1}{m} \right).$$

But

$$\pi \cot \pi \tau = \pi \frac{\cos \pi \tau}{\sin \pi \tau} = \pi i \frac{e^{2\pi i \tau} + 1}{e^{2\pi i \tau} - 1} = -\pi i \left(1 + 2 \sum_{r=1}^{\infty} q^r\right)$$

Hence

$$\frac{1}{\tau} + \sum_{m \in \mathbb{Z} - \{0\}} \left(\frac{1}{\tau + m} - \frac{1}{m} \right) = -\pi i \left(1 + 2 \sum_{r=1}^{\infty} q^r \right),$$

diff'ing

$$-1! \sum_{m \in \mathbb{Z}} \frac{1}{(\tau + m)^2} = -(2\pi i)^2 \sum_{r=1}^{\infty} r\, q^r$$

$$-3! \sum_{m \in \mathbb{Z}} \frac{1}{(\tau + m)^4} = -(2\pi i)^4 \sum_{r=1}^{\infty} r^3 q^r$$

$$\vdots$$

$\square$

Thm Again for each $k \in \{4, 6, 8, \dots\}$ we have

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\,q^n,$$

where $\sigma_r(n) := \sum_{d|n} d^r$.

Proof

Clearly

$$G_k(\tau) = 2\zeta(k) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}}\frac{1}{(m+n\tau)^k}.$$

But in (★) we may replace τ by $n\tau$ and sum over $n = 1, 2, \dots$

$$\sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(m + n\tau)^k} = \frac{1}{(k-1)!} (2\pi i)^k \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{nr}$$

$$= \frac{1}{(k-1)!} (2\pi i)^k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

and the thm follows $\square$

<u>Corollary</u> If we divide $G_k(\tau)$ by $2\zeta(k)$ we get

$$E_k(\tau) = 1 - \frac{B_k}{2k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is th k-th Bernoulli number,

$$\frac{x}{e^x - 1} =: \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

In particular,

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

$$\vdots$$

$$E_{12} = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$$

Remark The last one is closely related with the congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

In this notation we have

$$\Delta = \frac{E_4^{\,3} - E_6^{\,2}}{1728} \in S_{12}(\Gamma)$$

We may define *Klein's j-function*

$$j = \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + \underbrace{196884}_{\text{dimension of the Griess algebra } A} q + \dots ,$$

which is a modular function of weight 0.

$A \circlearrowleft \mathbb{M}$