§ 3 The space of modular forms

Let f be a meromorphic function on H, not identically zero, and let p be a point of H. The integer n s.t $\frac{f}{(z-p)^n}$ is holomorphic and nonzero at p is called the order of f at p and is denoted by $\nu_p(f)$.

If f is a modular function of weight $2k$, remember that we can identify these functions with som lattice functions of weight $2k$.

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

So, the order of f at any p is the same as the order at $g(p)$.

$$\nu_p(f) = \nu_{g(p)}(f) \qquad g \in G$$

In other words, $\nu_p(f)$ depends only in the image of p in H/G.

Remember A modular function is meromorphic at infinity, so we can express f as a meromorphic function $\tilde{f}$ at the origin.

If a modular function is holomorphic everywhere (including infinity) is called a modular form.

Define $\nu_\infty(f)$ as the order for $q=0$ of the function $\tilde{f}(q)$ associated to f.

Denote the stabilizer of p as e_p. $\quad I(p)=\{g \in G : g(p)=p\}$

We know by Thm 1: $I(z)=\{1\}$ except in the three cases.

$z=i$. The order of the group $I(z)$ is 2.

$z=\rho=e^{2\pi i/3}$ " " $\operatorname{ord}(I(z))=3$

$z=-\bar{\rho}=e^{\pi i/3}$ " " $\operatorname{ord}(I(z))=3$.

Theorem 3

Let f be a modular function of weight $2k$, not identically zero. One has

$$\nu_\infty(f) + \sum_{p \in H/G} \frac{\nu_p(f)}{e_p} = \frac{k}{6}$$

$$\left[\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) + \sum_{p \in H/G}^{*} \nu_p(f) = \frac{k}{6} \right.$$

$*$ sum over the distinct cla$\Big]$

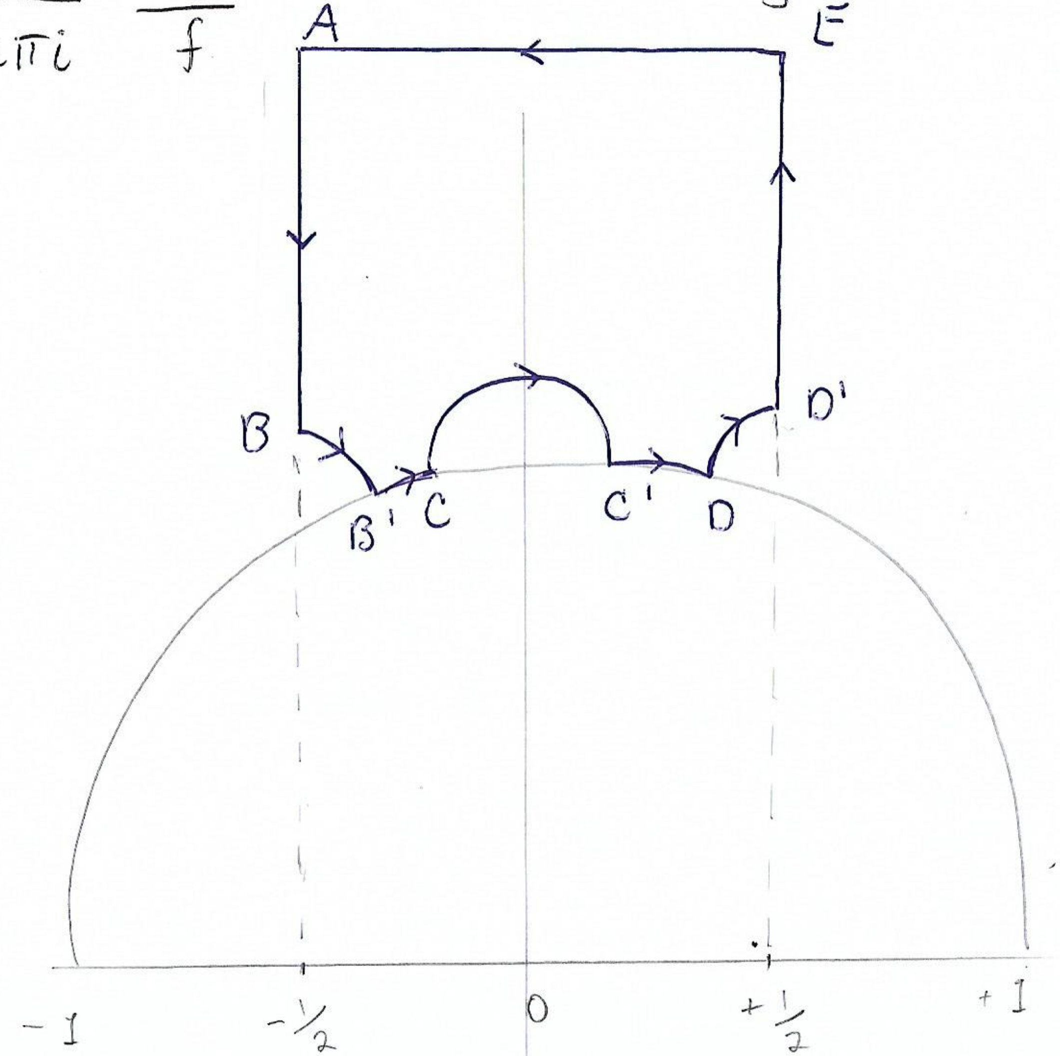
- The sum makes sense. The function f has only finite number of poles and zeros modulo G.
- Since $\tilde{f}$ is meromorphic, there is $r > 0$ s.t. $\hat{f}$ has no zero nor pole for $0 < |q| < r$. where $q = e^{2\pi i z}$

$$\Rightarrow \quad e^{-2\pi \operatorname{Im}(z)} < r$$

$$\Rightarrow \quad \frac{\log\left(\frac{1}{r}\right)}{2\pi} < \operatorname{Im}(z).$$

Now, the part D_r of the fundamental domain D defined by $\mathrm{Im}(z) \leq \frac{\log}{2\pi}$ is compact; since f is meromorphic in H, it has only a finite number of zeros and poles in D_r.

- Integrate $\frac{1}{2\pi i} \frac{df}{f}$ on the boundary of D.



$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{df}{f} = \int_{B'}^{B} + \int_{B'}^{C} + \int_{C'}^{C} + \int_{C'}^{D} + \int_{D}^{D'} + \int_{B}^{A} + \int_{D'}^{E} + \int_{A}^{E}$$

- Suppose that f has no zero nor pole on the boundary of D except for possibly i, ρ, $-\bar{\rho}$. That is why we can choose $\mathcal{C}$ as in the figure. That is, a curve whose interior contains a representative of each zero or pole of f not congruent to i or ρ. By R-T.

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{df}{f} = \sum_{p\in H/G}^{*} \nu_p(f)$$

- The change of variables $q = e^{2\pi i z}$ transforms the arc EA into a circle ω with center at 0, with negative orientation and that do not encloses any zero or pole of $\tilde{f}$ except possibly 0.

$$\frac{1}{2i\pi}\int_{E}^{A} \frac{df}{f} = \frac{1}{2\pi i}\int_{\omega} \frac{d\tilde{f}}{\tilde{f}} = -\nu_\infty(f)$$

- The integral of $\frac{1}{2\pi i}\frac{df}{f}$ on the circle which contains the arc BB', has value $-\nu_p(f)$. When the radius of the circle tends to 0, the angle BB' tends to $\frac{2\pi}{6}$, hence

$$\frac{1}{2\pi i}\int_B^{B'}\frac{df}{f} \longrightarrow -\frac{\nu_p(f)}{6}$$

In the same fashion, when the radius of the arcs CC' and DD' tend to 0:

$$\frac{1}{2\pi i}\int_C^{C'}\frac{df}{f} \longrightarrow -\frac{1}{2}\nu_i(f)$$

$$\frac{1}{2\pi i}\int_D^{D'}\frac{df}{f} \longrightarrow -\frac{1}{6}\nu_\rho(f)$$

- $T(z) = z+1$ transforms AB into ED', since $f(Tz) = f(z)$

$$\frac{1}{2\pi i}\int_A^B\frac{df}{f} + \frac{1}{2\pi i}\int_{D'}^{E}\frac{df}{f} = 0.$$

• $S(z) = \frac{-1}{z}$ transforms the arc $B'C$ onto the arc DC', since $f(Sz) = z^{2k} f(z)$, we get

$$\frac{df(Sz)}{f(Sz)} = 2k \frac{dz}{z} + \frac{df(z)}{f(z)}$$

hence:

$$\frac{1}{2\pi i}\int_{B'}^{C} \frac{df}{f} + \frac{1}{2\pi i}\int_{C'}^{D} \frac{df}{f} = \frac{1}{2\pi i}\int_{B'}^{C}\left(\frac{df(z)}{f(z)} - \frac{df(Sz)}{f(Sz)}\right)$$

$$= \frac{1}{2\pi i}\int_{B'}^{C}\left(-2k\frac{dz}{z}\right)$$

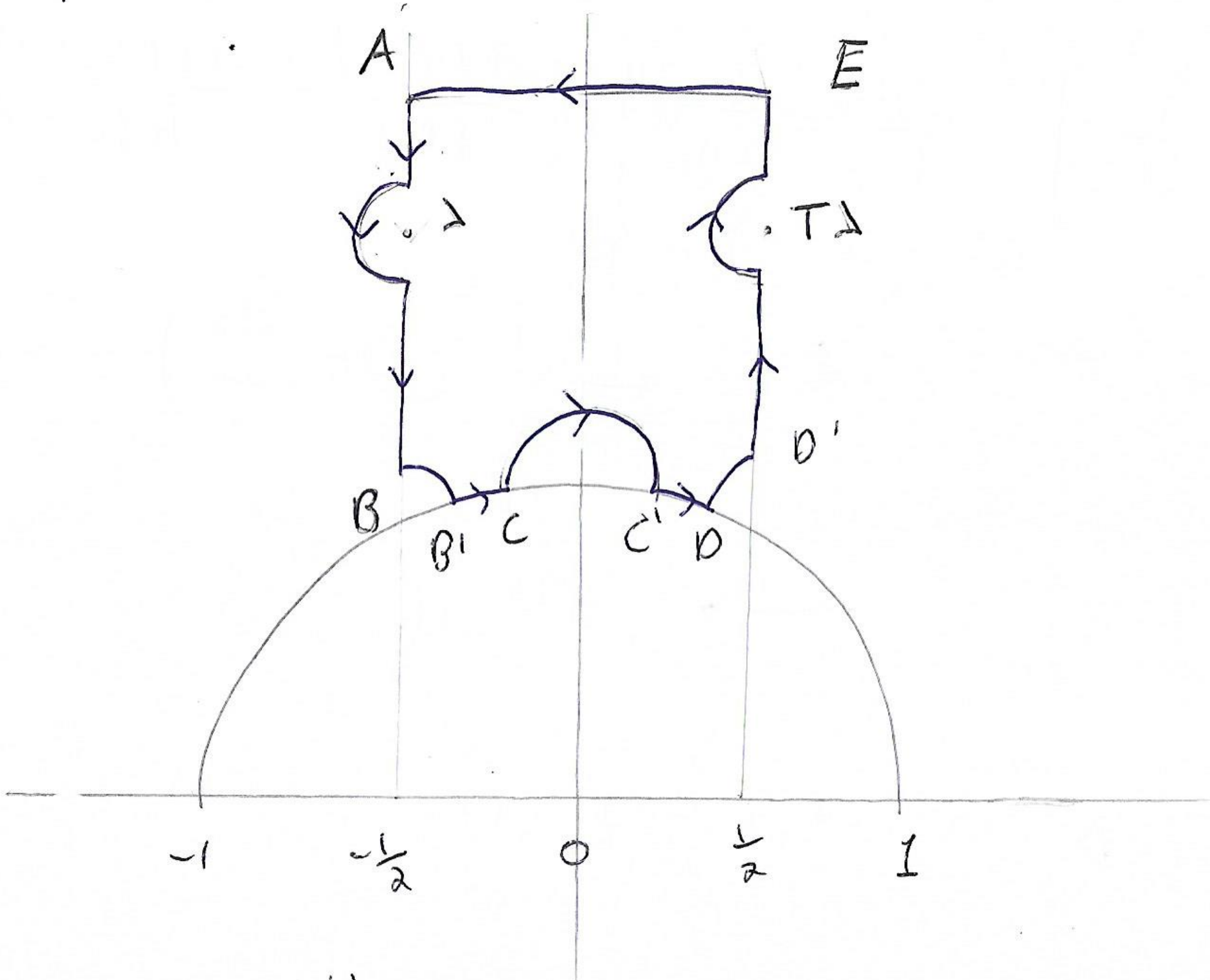
when $r \to 0$ $\longrightarrow -2k\left(\frac{-1}{12}\right) = \frac{k}{6}$

$$\Rightarrow \nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_e(f) + \sum_{p \in H/G}^{*} \frac{\nu_p(f)}{e_p} = \frac{k}{6}$$

- Suppose f has a zero or a pole λ on half of the line

$$\left\{ z : Re(z) = -\frac{1}{2},\ Im(z) > \frac{\sqrt{3}}{2} \right\}.$$

In that case repeat the proof with a contour as the following



Proceed analogously as in the previous case.