Lecture 28 On the valence formula

Given a non-zero meromorphic function f on $\mathcal{H}$ and a point $\tau_0 \in \mathcal{H}$ we define the order $\nu_{\tau_0}(f) \in \mathbb{Z}$ of f at τ_0 taking the Laurent expansion of f around τ_0

$$f(\tau) = a_{\nu_{\tau_0}(f)} (\tau - \tau_0)^{\nu_{\tau_0}(f)} + \dots$$

and insisting that $a_{\nu_0(f)} \in \mathbb{C}^\times$. If $f(\tau+1) = f(\tau)$, $\forall \tau \in \mathcal{H}$, the order $\nu_{i\infty}(f)$ of f at $i\infty$ by taking the Laurent expansion

$$f(\tau) = a_{\nu_\infty(f)}\, q^{\nu_\infty(f)} + \dots$$

similarily insisting that $a_{\nu_\infty(f)} \in \mathbb{C}^\times$. Here as before $q = e^{2\pi i \tau}$.

In order to make sense of the latter definition, consider $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ where $\mathbb{P}^1(\mathbb{Q})$ is the set of $\mathbb{Q}$-rational points of the projective line, and note that Γ acts on $\mathbb{P}^1(\mathbb{Q})$ if we let $\forall\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$g \cdot (\alpha : \beta) = (a\alpha + b\beta : c\alpha + d\beta),$$

and by identifying $\mathbb{P}^1(\mathbb{Q}) \cong \mathbb{Q} \cup \{\infty\}$, this action becomes

$$g \cdot r = \frac{ar+b}{cr+d}$$

Using Euclid's thm we see that $\Gamma \cdot \infty = \mathbb{P}^1(\mathbb{Q})$ and $\mathcal{H}^*/\Gamma$ is the (Alexandrov) one-point compactification of $\mathcal{H}/\Gamma$.

From now on let's assume that f is a modular function of weight k, so that $\forall\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\tau \in \mathcal{H}$,

$$f(g \cdot \tau) = (c\tau + d)^k f(\tau). \qquad \star$$

In particular, the order of function is well-defined on $\mathcal{H}/\Gamma$.

Let e_τ denote the order of the stabiliser of Γ at τ.

<u>Thm</u> (valence formula) If f is as above, then $\forall\ k \in \mathbb{Z}$

$$\sum_{\tau \in \mathcal{H}^*/\Gamma} \frac{1}{e_\tau} \nu_\tau(f) = \frac{k}{12}$$

Proof

The above follows from the complex-analytic structure of $\mathcal{H}^*/\Gamma$ of a Riemann surface of genus 0 together with the fact that

$$\omega := f(\tau)\, d\tau$$

is a k-fold (meromorphic) diff'l form on $\mathcal{H}^*/\Gamma$ $\square$

Lemma The function $\Delta := \dfrac{E_4^3 - E_6^2}{1728} \in S_{12}(\Gamma)$

does not vanish on $\mathcal{H}$ and has a simple zero at ∞.

Proof

As $\underset{\neq 0}{E_4} \in M_4(\Gamma)$, the valence formula says that for $n \in \mathbb{Z}_{\geq 0}$

$$v_\infty(E_4) + \frac{1}{2} v_i(E_4) + \frac{1}{3} v_\rho(E_4) + n = \frac{1}{3}.$$

Thus $v_\rho(E_4) = 1$, and $v_\tau(E_4) = 0$ for all $\tau \in \mathcal{H}$ s.t. τ is not Γ-equivalent to ρ. Similarly, as $\overset{\neq 0}{E_6} \in M_6(\Gamma)$, the valence formula

$$\nu_\infty(E_6) + \frac{1}{2}\nu_i(E_6) + \frac{1}{3}\nu_\rho(E_6) + n = \frac{1}{2}.$$

Thus $\nu_i(E_6) = 1$, and $\nu_\tau(E_6) = 0$ for all $\tau \in \mathcal{H}$ s.t. τ is not Γ-equivalent to i. In particular, $\Delta(i) \neq 0$.

Hence the valence formula for $\overset{\neq 0}{\Delta} \in S_{12}(\Gamma)$ says

$$\nu_\infty(\Delta) + \frac{1}{2}\nu_i(\Delta) + \frac{1}{3}\nu_\rho(\Delta) + n = 1,$$

Therefore $\nu_\infty(\Delta) = 1$ and $\nu_\tau(\Delta) = 0$, $\forall\, \tau \in \mathcal{H}$ $\square$

<u>Remark</u> Note that from the Fourier expansions of E_4 and E_6 we have, moreover, that the Spitzenform / cusp form Δ of weight $k = 12$ has a Fourier expansion of the form

$$\Delta = q + \ldots \qquad *$$

Later in our discussion on Hecke operators, we'll show that Δ is a normalised eigenform WRT the Hecke algebra $\mathbb{T}$ acting on $S_k(\Gamma)$; all such elements of $S_k(\Gamma)$ start like $(*)$.

Lemma We have a $\mathbb{C}$-vector space isomorphism

$$M_{k-12}(\Gamma) \xrightarrow{\Psi} S_k(\Gamma)$$

$$f \longmapsto \Delta f$$

Proof

From the previous lemma, $\forall\, h \in S_k(\Gamma)$ the function $f := \Delta^{-1} h$ is s.t.

$$\nu_\tau(f) = \begin{cases} \nu_\tau(h), & \text{if } \tau \neq \infty \\ \nu_\tau(h) - 1, & \text{if } \tau = \infty \end{cases}$$

Thus $f \in M(\Gamma)$ and the lemma follows $\square$

Thm (Dimension formula) For each $k \in 2\mathbb{Z}_{\geq 0}$

$$\dim_{\mathbb{C}}(M_k(\Gamma)) = \begin{cases} \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1, & \text{if } k \not\equiv 2 \pmod{12} \end{cases}$$

Proof

From the valence formula we have for $k = 2$ and for $k \in \mathbb{Z}_{<0}$, $M_k(\Gamma) = 0$. So the previous lemma gives

$$\dim_{\mathbb{C}}(M_k(\Gamma)) \leq 1,$$

if $k \in \{0, 4, 6, 8, 10, 14\}$. But $E_k^{\neq 0} \in M_k(\Gamma)$. Thus the dimension of $M_k(\Gamma)$ is

1 2 3 ...

$$M_0 = \mathbb{C}\,1 \hookrightarrow M_{12} = S_{12} \oplus \mathbb{C}E_{12} \hookrightarrow M_{24} = S_{24} \oplus \mathbb{C}E_{24} \hookrightarrow \dots$$

$$M_4 = \mathbb{C}E_4 \hookrightarrow M_{16} = S_{16} \oplus \mathbb{C}E_{16} \hookrightarrow M_{28} = S_{28} \oplus \mathbb{C}E_{28} \hookrightarrow \dots$$

$$M_6 = \mathbb{C}E_6 \hookrightarrow M_{18} = S_{18} \oplus \mathbb{C}E_{18} \hookrightarrow M_{30} = S_{30} \oplus \mathbb{C}E_{30} \hookrightarrow \dots$$

$$M_8 = \mathbb{C}E_8 \hookrightarrow M_{20} = S_{20} \oplus \mathbb{C}E_{20} \hookrightarrow M_{32} = S_{32} \oplus \mathbb{C}E_{32} \hookrightarrow \dots$$

$$M_{10} = \mathbb{C}E_{10} \hookrightarrow M_{22} = S_{22} \oplus \mathbb{C}E_{22} \hookrightarrow M_{34} = S_{34} \oplus \mathbb{C}E_{34} \hookrightarrow \dots$$

$$M_{14} = \mathbb{C}E_{14} \hookrightarrow M_{26} = S_{26} \oplus \mathbb{C}E_{26} \hookrightarrow M_{38} = S_{38} \oplus \mathbb{C}E_{38} \hookrightarrow \dots$$

$\square$

Put $X := \mathcal{H}^*/\Gamma$, equipped with its complex-analytic structure.

<u>Thm</u> We have

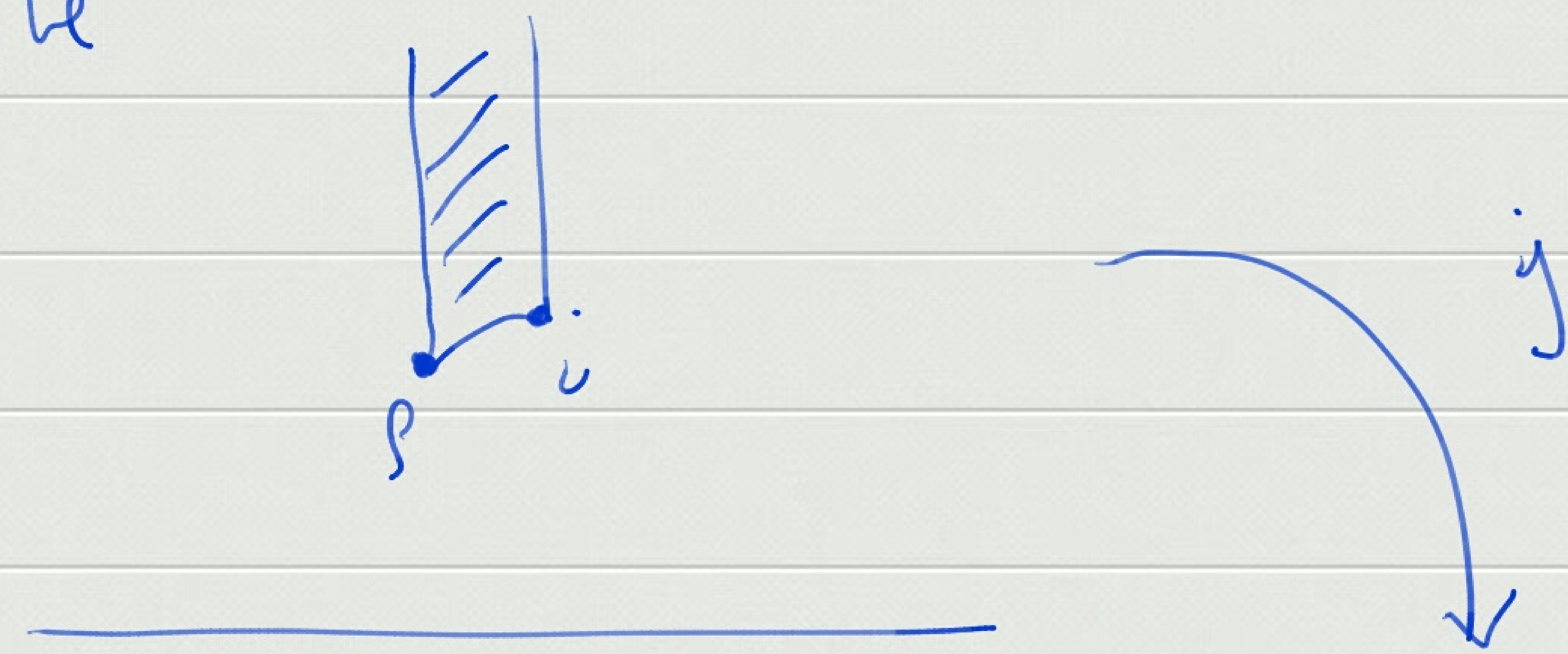
$$\mathcal{M}(X) = \mathbb{C}(j),$$

where $\mathcal{M}(X)$ denotes the field of meromorphic functions on X and $\mathbb{C}(j)$ is the field of rational functions on Klein's j-function.

Proof

[Klar.]

In particular we may see that $j(\tau)$ maps the hyperbolic triangle



to the upper half plane: