Analytic methods
Modular forms
Theta functions

1.

6.1 The Poisson formula

Let V be a real vector space of finite dimension n endowed with an invariant measure μ.

2. Let V' be the dual of V. Let f be a rapidly decreasing smooth function on V.
The fourier transform f' of f is defined by the formula

$$f'(y) = \int_V e^{-2\pi i \langle x,y \rangle} f(x)\,\mu(x).$$

This is a rapidly decreasing smooth function on V'.

3 The Fourier transform f' of f is defined by the formula

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, which is a rapidly decreasing smooth function on V'.

Let now Γ be a lattice in V'. We denote by Γ' the lattice in V' dual to Γ; it is the set of $y \in V'$ such that $\langle x,y \rangle \in \mathbb{Z}$ for all $x \in \Gamma$.

4. **Proposition 15.** Let $v = \mu(V/\Gamma)$. One has:

$$\sum_{x \in \Gamma} f(x) = \frac{1}{v} \sum_{y \in \Gamma'} f'(y).$$

After replacing μ by $v^{-1}\mu$, we can assume that $\mu(V/\Gamma) = 1$. By taking a basis $e_1, \dots, e_n$ of Γ, we identify V with $\mathbb{R}^n$, Γ with $\mathbb{Z}^n$, and μ with the product measure $dx_1 \dots dx_n$. Thus we have $V' = \mathbb{R}^n$, $\Gamma' = \mathbb{Z}^n$ and we are reduced to the classical Poisson formula.

5.

Let $f:\mathbb{R}\to\mathbb{C}$ be a function

f is a Schwartz function if and only if :

$$\forall c\in\mathbb{R},\ n\in\mathbb{N}_0 : |f^{(n)}(x)| = o(|x|^c)$$ where $f^{(n)}$ denotes the nth derivative

Theorem

Let $f:\mathbb{R}\to\mathbb{C}$ be a Schwartz function.

Let $\hat{f}$ be its Fourier transform

then $$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$$

Proof.

Let $$F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

then $F(x)$ is 1-periodic (because of absolute convergence), and has Fourier coefficients :

$$\hat{F}_k = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)\, e^{-2\pi i k x}\,dx = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n)\, e^{-2\pi i k x}\,dx$$ because f is Schwartz, so convergence is uniform.

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)\, e^{-2\pi i k x}\,dx = \int_{\mathbb{R}} f(x)\, e^{-2\pi i k x}\,dx = \hat{f}(k)$$ where $\hat{f}$ is the Fourier transform of f.

Therefore by the definition of the Fourier series of F :

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\, e^{ikx}$$. Choosing $x=\phi$ in this formula :

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$ as required.

6.2 Application to quadratic forms

We suppose henceforth that V is endowed with a symmetric bilinear form $x.y$ which is positive and nondegenerate (i.e. $x.x>0$ if $x\neq 0$).

6. We identify V with V' by means of this bilinear form. The lattice Γ' becomes thus a lattice in V; one has $y\in\Gamma'$ if and only if $x.y\in\mathbb{Z}$ for all $x\in\Gamma$.

7. To a lattice Γ, we associate the following function defined on $\mathbb{R}^*_+$

$$\Theta_\Gamma(t)=\sum_{x\in\Gamma} e^{-\pi t\, x.x}$$

We choose the invariant measure μ on V such that, if $\varepsilon_1,\dots,\varepsilon_n$ is an orthonormal basis of V, the unit cube defined by the ε_i has volume 1. The volume v of the lattice Γ is then defined by $v=\mu(V/\Gamma)$.

8.

Proposition 16. We have the identity

$$\Theta_\Gamma(t)=t^{-n/2}v^{-1}\Theta_{\Gamma'}(t^{-1})$$

Let $f=e^{-\pi x.x}$. It is a rapidly decreasing smooth function on V.
We choose an orthonormal basis of V and use this basis to identify V with $\mathbb{R}^n$; the measure μ becomes the measure $dx=dx_1\dots dx_n$ and the function f is $f=e^{-\pi(x_1^2+\dots+x_n^2)}$. Using $x=(x_1,\dots,x_n)$.

9. We are thus reduced to showing that the Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi x^2}$
We are going to show this:
Let $f(x)=e^{-\pi x^2}$
By the definition of the Fourier transform we see that

$$F(s)=\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-j2\pi s x}dx=\int_{-\infty}^{\infty} e^{-\pi(x^2+j2sx)}dx$$

Now we can multiply the right handside by $e^{-\pi s^2}e^{\pi s^2}=1$:

$$F(s)=e^{-\pi s^2}\int_{-\infty}^{\infty} e^{-\pi(x^2+j2sx)+\pi s^2}dx=e^{-\pi s^2}\int_{-\infty}^{\infty} e^{-\pi(x^2+j2sx-s^2)}dx$$

$$=e^{-\pi s^2}\int_{-\infty}^{\infty} e^{-\pi(x+js)(x+js)}dx=e^{-\pi s^2}\int_{-\infty}^{\infty} e^{-\pi(x+js)^2}dx$$

$$F(s)=e^{-\pi s^2}\int_{-\infty}^{\infty} e^{-\pi(x+js)^2}dx$$

After substituting u for $x+is$ and du for dx we see that

$$F(s) = e^{-\pi s^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2} du}_{1}.$$

It follow that the Gaussian is its own Fourier Transform.

$$e^{-\pi x^2} \xrightarrow{F} e^{-\pi s^2}.$$

We now apply Prop.15 to the function f and to the lattice $t^{1/2}\Gamma$; the volume of the lattice is $t^{n/2}v$ and its dual is $t^{-1/2}\Gamma'$; hence we get the formula to be proved.

$$\sum_{x\in\Gamma} e^{-\pi x^2} = \frac{1}{t^{n/2}v} \sum_{y\in\Gamma'} e^{-\pi x^2}$$

6.3 Matrix interpretation

let $e_1, \dots, e_n$ be a basis of Γ. Put $a_{ij} = e_i e_j$. The matrix $A = (a_{ij})$ is positive, nondegenerate and symmetric. If $x = \sum x_i e_i$ is an element of V, then $x.x = \sum a_{ij} x_i x_j$

10. the function Θ_Γ can be written :

$$\Theta_\Gamma(t) = \sum_{x_i \in \mathbb{Z}} e^{-\pi t \sum a_{ij} x_i x_j},$$

the volume v of Γ is given by:

$$v = \det(A)^{1/2}$$

This can be seen as follows.

let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal basis of V and put

$$\varepsilon = \varepsilon_1 \wedge \dots \wedge \varepsilon_n \quad , e = e_1 \wedge \dots \wedge e_n$$

11. We have $e = \lambda \varepsilon$ with $|\lambda| = v$. Moreover, $e.e = \det A\, \varepsilon.\varepsilon$, and by comparing, we obtain $v^2 = \det(A)$.

12. let $B = (b_{ij})$ be the matrix inverse to A. One checks immediately that the dual basis (e_i') to (e_i) is given by the formula: $b_{ij} = e$

$$e_i' = \sum b_{ij} e_j. \qquad e_i \sum b_{ij} e_j = \sum b_{ij} a_{ij}$$

13. the (e_i') form a basis of Γ'. The matrix $(e_i' \cdot e_j')$ is equal to B. This shows in particular that if $v' = \mu(V/\Gamma')$, then we have $vv' = 1$

14.

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Special case

We will be interested in pairs (V,Γ) which have the following 2 properties:
(i) the dual Γ' of Γ is equal to Γ

This amounts to saying that one has $x.y \in \mathbb{Z}$ for $x,y \in \Gamma$ and that the form $x.y$ defines an isomorphism of Γ onto its dual.

1. In matrix terms, it means that the matrix $A=(e_i \cdot e_j)$ has integer coefficients and that its determinant equals 1.

2. As we know the volume V of the lattice Γ is given by $v = \det(A)^{1/2}$, and as the determinant is 1, then $v=1$.

If $n = \dim V$, this condition implies that the quadratic module Γ belongs to the category S_n defined in n° 1.1 of chapter V.

3. Conversely, if $\Gamma \in S_n$ is positive definite, and if one puts $V = \Gamma \otimes \mathbb{R}$, the pair (V,Γ) satisfies (i).

4. We have $x.x \equiv 0 \pmod 2$ for all $x \in \Gamma$

this means that Γ is of type II, in the sense that the diagonal terms $e_i \cdot e_i$ of the matrix A are even.

5. Example:

- We denote by U the element of S_2 defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The associated quadratic form is $2x_1x_2$; U is of type 2. One has:

$$r(U)=2, \quad \tau(U)=0, \quad d(U)=-1, \quad \sigma(U)=0$$

6. Theta functions.

In this section and the next one, we assume that the pair (V,Γ) satisfies conditions (i) and (ii) of the preceding section.

Let m be an integer ≥ 0, and denote by $r_\Gamma(m)$ the number of elements x of Γ such that $x.x = 2m$. It is easy to see that $r_\Gamma(m)$ is bounded by a polynomial in m (a crude volume argument gives for instance $r_\Gamma(m) = O(m^{n/2})$.

7. this shows that the series with integer coefficients

$$\sum_{m=0}^{\infty} r_\Gamma(m)\, q^m = 1 + r_\Gamma(1)q + \dots$$

converges for $|q|<1$.

geometric series and $r_\Gamma(m)$ is an integer.

Thus one can define a function Θ_Γ on the half plane H by the formula:

$$\Theta_\Gamma(z) = \sum_{m \geq \phi} r_\Gamma(m)\, q^m \quad (\text{where } q = e^{2\pi i z})$$

We have:

$$\Theta_\Gamma(z) = \sum_{x \in \Gamma} q^{(x,x)/2} = \sum_{x \in \Gamma} e^{\pi i z (x,x)}$$

The function Θ_Γ is called the theta function of the quadratic module Γ. it is holomorphic on H.

8. Theorem 8.

(a) The dimension n of V is divisible by 8.

(b) The function Θ_Γ is a modular form of weight $n/2$.

We first prove the identity:

$$\Theta_\Gamma(-1/z) = (iz)^{n/2}\, \Theta_\Gamma(z)$$

Since the 2 sides are analytic in z, it suffices to prove this formula when $z = it$ with t real $> \phi$.

9. We have:

$$\Theta_\Gamma(it) = \sum_{x \in \Gamma} e^{-\pi t (x,x)} = \Theta_\Gamma(t) \quad \Big\}\ \text{By definition}$$

Similarly, $\Theta_\Gamma(-1/it) = \Theta_\Gamma(t^{-1}) = \sum_{x \in \Gamma} e^{\frac{-\pi i}{it}(x,x)} = \sum_{x \in \Gamma} e^{-\pi t^{-1}(x,x)}$

Using proposition 16, which says that:

$$\Theta_\Gamma(t) = t^{-n/2}\, v^{-1}\, \Theta_{\Gamma'}(t^{-1})$$

and taking into account that $v = 1$ and $\Gamma = \Gamma'$.

then $$\Theta_\Gamma(t) = t^{-n/2}\, \Theta_\Gamma(t^{-1})$$

And then $$\Theta_\Gamma(-1/z) = (iz)^{n/2}\, \Theta_\Gamma(z).$$ ∎

Proof.

(a). Suppose that n is not divisible by 8; replacing Γ, if necessary by $\Gamma \oplus \Gamma$ or $\Gamma \oplus \Gamma \oplus \Gamma \oplus \Gamma$, we may suppose that $n \equiv 4 \pmod 8$

10. The Formula $\Theta_\Gamma(-1/z) = (iz)^{n/2}\, \Theta_\Gamma(z)$ can then be written

$$\Theta_\Gamma(-1/z) = (-1)^{n/4}\, z^{n/2}\, \Theta_\Gamma(z) = -z^{n/2}\, \Theta_\Gamma(z)$$

This is because as $n \equiv 4 \pmod 8$, then n can be seen as: $n = 8k+4$; $k = 1,2,3,\ldots$
then the identity we proved before is going to become:

$$\Theta_\Gamma(-1/z) = (iz)^{n/2}\Theta_\Gamma(z) = i^{n/2}\, z^{n/2}\,\Theta_\Gamma(z) = (i^4)^k \cdot i^2\, z^{n/2}\,\Theta_\Gamma(z)$$

$$= -z^{n/2}\,\Theta_\Gamma(z)$$

If we put $\omega(z) = \Theta_\Gamma(z)\, dz^{n/4}$, we see that the differential form ω is transformed into $-\omega$ by $S: z \longrightarrow \frac{-1}{z}$.

11. We are going to see this:

$$\omega(z) = \Theta_\Gamma(z)\, d z^{n/4}$$

$$S: z \longrightarrow \frac{-1}{z}$$

$$\Theta_\Gamma\left(\frac{-1}{z}\right) d\left(\frac{-1}{z}\right)^{n/4}$$

$$\Theta_\Gamma\left(-\frac{1}{z}\right) d(-z^{-1})^{n/4}$$

, we know that $\Theta_\Gamma\left(-\frac{1}{z}\right) = -z^{n/2}\,\Theta_\Gamma(z)$

then this becomes $-z^{n/2}\,\Theta_\Gamma(z)\, d(-z^{-1})^{n/4} = +z^{n/2}\,\Theta_\Gamma(z)\, d(z^{-n/4})$

Since ω is invariant by $T: z \longrightarrow z+1$, we see that ST transforms ω into $-\omega$, which is absurd because $(ST)^3 = 1$.

∎

Corolary 1 There exists a cusp form f_Γ of weight $\frac{n}{2}$ such that
$$\Theta_\Gamma = E_k + f_\Gamma \quad \text{where } k = \frac{n}{4}.$$

1. **Definition 1.** We say that f is meromorphic at infinity (respectively holomorphic at infinity) if $f(z) = \sum_{n \geq n_0} a(n) q^n$ (respectively if in addition $n_0 = 0$).

where $a(n)$: denotes the number of elements x of Γ such that $x.x = 2n$ for an integer $n \geq \phi$, recall that $a(n) = O(n^{m/2})$.
And $q = e^{2\pi i z}$.

Note that checking that f is holomorphic at infinity is the same as checking that $f(z)$ is bounded as z aproaches $i\infty$. If f is holomorphic at infinity, then the value of f at infinity is defined to be $f(\infty) = a(0)$.

Definition 2. We say that f vanishes at infinity if $n_0 = 1$. Equivalently, if $f(\infty) = \phi$

Definition 3. let $k \in \mathbb{Z}$ and let $f : H \to \mathbb{C}$. We say that f is a modular form of weight k for $Sl_2(\mathbb{Z})$ if for $k \in \mathbb{Z}$

1. f is holomorphic
2. $f(\gamma z) = (cz + d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z})$, and
3. f is holomorphic at infinity

A cusp form is a modular form which vanishes at infinity.

2. By theorem 8, Θ_Γ is a modular form of weight $\frac{n}{2}$ and is simply to see that $\Theta_\Gamma(\infty) = 1$.

3. We also want to prove $E_k(z) = \dfrac{G_k(z)}{2\zeta(2k)}$ is a modular form of weight $\frac{n}{2}$ where $k = \frac{n}{4}$ and that the function evaluated in ∞ gives 1.

By proposition 4 of this book we have that for an integer $k > 1$ the Eisenstein series $G_k(z)$ is a modular form of weight $2k$ and that $G_k(\infty) = 2\zeta(2k)$.
So if $k = \frac{n}{4}$, then $G_k(z)$ is a modular form of weight $\frac{n}{2}$ and
$$\frac{G_k(\infty)}{2\zeta(2k)} = E_k(\infty) = 1.$$
Which is what we wanted to prove.
$\Longrightarrow \Theta_n - E_k$ is a cusp form. ■

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   $\longrightarrow \Theta_\Gamma - E_k$ is a cusp form.

Corollary 2. We have $r_\Gamma(m) = \frac{4k}{B_k}\sigma_{2k-1}(m) + O(m^k)$ where $k = \frac{n}{4}$

We know that $r_\Gamma(m)$ denote the number of elements x of Γ such that $x.x = 2m$ and where $\Theta_\Gamma(z) = \sum_{m \in \varphi} r_\Gamma(m) q^m$ (where $q = e^{2\pi i z}$)

We also know by formula (34) that

$$E_k(z) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

In conclusion, the coefficient of the modular form of $\Theta_\Gamma(z)$ is $r_\Gamma(m)$ and the coefficient of the modular form of $E_k(z)$ is $\frac{4k}{B_k}\sigma_{2k-1}$

Now, with this in count we can proceed.

Let a_m denote the coefficient of the modular form of f_Γ

As $f_\Gamma = \Theta_\Gamma - E_k$, by corollary 1,

$\Rightarrow$ a_m is also the coefficient of the modular form of $\Theta_\Gamma - E_k$

Using the results above we have that $a_m = r_\Gamma(m) - \frac{4k}{B_k}\sigma_{2k-1}$

By corollary 1, f_Γ is a cusp of weight $\frac{n}{2}$ and by theorem 5 (Hecke), we know that $a_m = O(m^k)$

$$\Rightarrow \quad a_m = r_\Gamma(m) - \frac{4k}{B_k}\sigma_{2k-1} = O(m^k)$$

$$\Rightarrow \quad r_\Gamma(m) = \frac{4k}{B_k}\sigma_{2k-1}(m) + O(m^k) \quad \text{where } k = \frac{n}{4}.$$