Lecture 32 Can we hear the shape of a 16-dimensional torus?

For each $n \in \{1, 2, 3, \ldots\}$ consider the second-order elliptic operator

$$\begin{aligned} \mathcal{L}^2(\mathbb{R}^n) &\longrightarrow \mathcal{L}^2(\mathbb{R}^n) \\ f &\longmapsto \Delta f := -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}. \end{aligned}$$

This is known as the Laplacian on $\mathbb{R}^n$. For each lattice $L \subseteq \mathbb{R}^n$, we may define the Laplacian on the torus $\mathbb{R}^n/L$ by restricting to the space of L-invariant functions $\mathcal{L}^2(\mathbb{R}^n/L)$.

We say that $f \in \mathcal{L}^2(\mathbb{R}^2/L)$ is an eigenfunction of Δ if $\exists\, \lambda \in \mathbb{C}$ s.t. $\Delta f = \lambda f$ and the corresponding set of eigenvalues is known as the spectrum $\mathcal{S}$ of Δ on $\mathbb{R}^2/L$. It is known that $\mathcal{S} \subseteq \mathbb{R}_{\geqslant 0}$ and, moreover, we have the following.

<u>Lemma</u> For each lattice $L \subseteq \mathbb{R}^2$ the spectrum of Δ on $\mathbb{R}^2/L$ is given by

$$\mathcal{S} = \{ \lambda_\omega \mid \omega \in L \}$$

where

$$\lambda_\omega := 4\pi^2 |\omega|^2.$$

Proof

Given a lattice $L \subseteq \mathbb{R}^n$ and a fixed $\omega \in L$, we have a well-defined function

$$\begin{aligned} \mathbb{R}^n / L &\xrightarrow{f_\omega} \mathbb{C} \\ v + L &\longmapsto e^{2\pi i\, \omega \cdot v} \end{aligned}$$

s.t. $\Delta f_\omega = \lambda_\omega f_\omega$, where the eigenvalue $\lambda_\omega = 4\pi^2 |\omega|^2$ has multiplicity given by the # of $\omega' \in L$ on the sphere of radius $\frac{\sqrt{\lambda_\omega}}{2\pi}$.

But $\{ f_\omega \mid \omega \in L \} \subseteq \mathcal{L}^2(\mathbb{R}^n / L)$ is dense and the lemma follows $\square$

Thm (Milnor 1964) There are lattices $L_1, L_2 \subseteq \mathbb{R}^{16}$ s.t. the corresponding tori $\mathbb{R}^{16}/L_1$ and $\mathbb{R}^{16}/L_2$ have the same spectrum, but they are not isometric.

Proof

There exist precisely two classes of isomorphism of unimodular lattices L_1, L_2 of rank 16, and thus <u>non-isometric</u> tori $\mathbb{R}^{16}/L_1$ and $\mathbb{R}^{16}/L_2$, as shown in

Witt, E., *Eine Identität zwischen Modulformen zweiten Grades*, Abhanglungen aus dem Mathematischen Seminar der Universität Hamburg, 14, pp. 323 – 337 (1941).

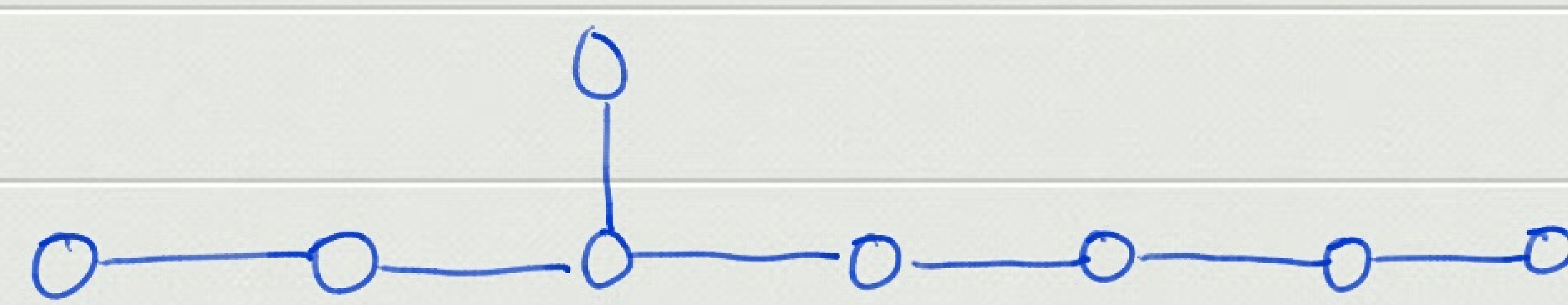
As L_1 and L_2 are unimodular of rank 16, the corresponding theta functions $\vartheta_{L_1}(\tau)$ and $\vartheta_{L_2}(\tau)$ both are modular forms (for $\Gamma = PSL_2(\mathbb{Z})$) of weight $k=8$. But the vector space $M_8(\Gamma)$ is 1-dimensional.

Therefore

$$\vartheta_{L_1}(\tau) = \vartheta_{L_2}(\tau) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)\, q^n,$$

where $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$. So the lemma implies that $\mathbb{R}^{16}/L_1$ and $\mathbb{R}^{16}/L_2$ have the same spectrum, i.e. Milnor's thm follows $\square$

Remark The above lattices found by Witt are now usually called $E_8 \oplus E_8$ and D_{16}^+, where E_8 is the root lattice generated by the root system with Dynkin diagram



and D_n^+ is the n-dimensional diamond packing[1].

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1 The points of D_3^+ correspond to the positions of carbon atoms in a diamond.

The reader is adviced to get

Conway J., The Sensual (quadratic) Form, The Carus Mathematical Monographs, 26, The Mathematical Association of America.